Commutative W*-algebras as a Markov Category (Extended Abstract)

Robert Furber

July 27, 2023

Abstract

We show that the category $\mathbf{CW^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$, the opposite of the category of commutative W*-algebras with positive unital maps as morphisms, is a Markov category. We do this by showing that the comonad on $\mathbf{CW^*Alg}$ of which $\mathbf{CW^*Alg}_{\mathrm{PU}}$ is the coKleisli category is commutative, where the chosen tensor product is the coproduct (of $\mathbf{CW^*Alg}$, since $\mathbf{CW^*Alg}_{\mathrm{PU}}$ doesn't have one). It follows, by the duality between commutative W*-algebras and measure spaces, that the corresponding monad on the category of compact complete strictly localizable measure spaces is commutative.

On the way, we give a universal property in $\mathbf{CW^*Alg}_{\mathrm{PU}}$ for the colimits of $\mathbf{CW^*Alg}$ in terms of "true continuity" of positive-operator-valued measures. This is essential to describe the positive unital maps out of a $\mathbf{CW^*Alg}$ coproduct. We can then explicitly calculate the coproduct $L^\infty([0,1]) + L^\infty([0,1])$ as $L^\infty([0,1])^{2^{\aleph_0}}$.

This is an extended abstract of the preprint [1].

In [2] the author showed that the category $\mathbf{CW^*Alg}_{\mathrm{PU}}$ having commutative W*-algebras[1] as objects and normal[2] positive unital linear maps as morphisms is the coKleisli category of a comonad H on $\mathbf{CW^*Alg}$ (with morphisms normal unital *-homomorphisms).

This provided a "probabilistic Gelfand duality" analogous to that of [3], but for the Gelfand duality between measure spaces and commutative W*-algebras [4] instead of compact Hausdorff spaces and commutative unital C*-algebras. The advantage of measure spaces and W*-algebras is that (normal) conditional expectations always exist, whereas conditional expectations for C*-algebras can fail to exist for topological reasons, which hinders the development of the theory of conditional probability.

The probability monad $\mathcal{R}$ used in [3] is commutative in the sense of [5, Corollary 3.7]: there is a map $\nabla_{X,Y} : \mathcal{R}(X) \times \mathcal{R}(Y) \to \mathcal{R}(X \times Y)$ satisfying certain conditions. Specifically, this map takes two Radon probability measures to their independent product. So it is natural to ask if $H : \mathbf{CW^*Alg} \to \mathbf{CW^*Alg}$ is, *i.e.* if we have the required map $\nabla_{A,B} : H(A + B) \to H(A) + H(B)$, to represent independent product measures (in dual form). This would prove that $\mathbf{CW^*Alg}_{\mathrm{PU}}$ is a Markov category [6, §3].

[1] Sometimes known as von Neumann algebras, strictly speaking these are only the same up to isomorphism and $L^\infty(X, \Sigma_X, \nu_X)$ is not literally a von Neumann algebra.

[2] Equivalently weak-* continuous.

In order to do this, we prove a characterization of $\mathbf{CW^*Alg}_{\mathrm{PU}}$ morphisms out of $\mathbf{CW^*Alg}$-colimits. We first observe that for (X, Σ_X, ν_X) a localizable measure space, the morphisms $\mathbf{CW^*Alg}_{\mathrm{PU}}(L^\infty(X), B)$ can be viewed as positive-operator-valued measures $\xi : \Sigma_X \to B_+$ (known as POVMs for short) that are *truly continuous*[3] to ν_X. We notate "ξ is truly continuous to ν_X" as $\xi \lll \nu_X$. For a compact Hausdorff space X, and a W*-algebra B we write $\mathbf{POVM}(X; B)$ for the set of B-valued POVMs on the Baire σ-algebra of X.

Consider a diagram $\mathcal{D} \to \mathbf{CHaus}$, notated as $(X_i)_{i\in\mathcal{D}}$, such that each X_i is equipped with a localizable Baire measure ν_{X_i} and the morphisms are normal morphisms of measure spaces. We can take the limit of this diagram in $\mathbf{CHaus}$, the usual closed subspace of the product. Applying the functor L^∞ from localizable measure spaces to $\mathbf{CW^*Alg}$, we also obtain a diagram $(L^\infty(X_i))_{i\in\mathcal{D}}$ in $\mathbf{CW^*Alg}$. Given a commutative W*-algebra B, we define the POVMs with *Truly Continuous Marginals* $\mathbf{TCMPOVM}((X_i)_{i\in\mathcal{D}}; B)$ to be

$$\begin{aligned}&\mathbf{TCMPOVM}((X_i)_{i\in\mathcal{D}}; B)\\ &= \{\xi \in \mathbf{POVM}(\lim_{i\in\mathcal{D}} X_i; B) \mid \forall i \in \mathcal{D}.(\pi_i)_*(\xi) \lll \nu_{X_i}\},\end{aligned}$$

where $\pi_i : \lim_{i\in\mathcal{D}} X_i \to X_i$ are the projection maps forming the limiting cone, and $(\text{-})_*$ is the operation of pushing a POVM along a measurable map (hence taking the marginal POVM).

Then the characterization is

$$\mathbf{CW^*Alg}_{\mathrm{PU}}(\operatorname*{colim}_{i\in\mathcal{D}} L^\infty(X_i); B) \cong \mathbf{TCMPOVM}((X_i)_{i\in\mathcal{D}}; B),$$

as a natural isomorphism with respect to the B argument. In the specific case of binary coproducts, we have

$$\begin{aligned}&\mathbf{CW^*Alg}_{\mathrm{PU}}(L^\infty(X) + L^\infty(Y); B)\\ &\cong \{\xi \in \mathbf{POVM}(X \times Y; B) \mid (\pi_1)_*(\xi) \lll \nu_X \text{ and } (\pi_2)_*(\xi) \lll \nu_Y\}.\end{aligned}$$

Further specializing to $B = \mathbb{C}$ shows us that the normal states on $L^\infty(X) + L^\infty(Y)$ are given by the Baire (isomorphically, Radon) probability measures on $X \times Y$ whose respective marginals are truly continuous to ν_X and ν_Y, a fact anticipated by Dauns in [8, Definition 2.5, paragraph starting "Alternatively", and 4.8 Theorem I (ii)].

By constructing a continuum-sized mutually singular family of measures on $2^\omega \times 2^\omega$ whose marginals on each side are the usual independent fair-coin-flipping measure ν_c on 2^ω, we are able to prove that $L^\infty(2^\omega, \nu_c) + L^\infty(2^\omega, \nu_c) \cong L^\infty(2^\omega, \nu_c)^{2^{\aleph_0}}$. Under a standard isomorphism, this proves the same fact for $L^\infty([0, 1])$ using the Lebesgue measure.

Using the characterization above, we can finally define the *-homomorphism $\nabla_{X,Y} : H(L^\infty(X) + L^\infty(Y)) \to H(L^\infty(X)) + H(L^\infty(Y))$ as a POVM with truly continuous marginals, and then extend the definition from $L^\infty(X)$ of a Baire measure to all commutative W*-algebras using Gelfand duality (the hyperstonean spaces formulation). After proving the relevant diagrams commute, this shows that $\mathbf{CW^*Alg}_{\mathrm{PU}}{}^{\mathrm{op}} \simeq \mathcal{K}\ell(H)^{\mathrm{op}}$ is a Markov category.

[3]This is a stronger condition than absolute continuity and necessary in the non-σ-finite case, equivalent in the σ-finite case. It is an extension of Fremlin's definition [7, 232A(b), 327C(e)].

References

[1] Robert Furber. Commutative W*-algebras as a Markov Category (Preprint). `https://robertfurber.com/preprints/markcat-preprint.pdf`, 2023.

[2] Robert Furber. A Probability Monad on Measure Spaces (Preprint). `https://robertfurber.com/preprints/cemonad-preprint.pdf`, 2022.

[3] Robert Furber and Bart Jacobs. From Kleisli Categories to Commutative C*-algebras: Probabilistic Gelfand Duality. *Logical Methods in Computer Science*, 11(2):1–28, 2015. doi:10.2168/LMCS-11(2:5)2015.

[4] Dmitri Pavlov. Gelfand-type duality for commutative von Neumann algebras. *Journal of Pure and Applied Algebra*, 226(4):106884, 2022. ISSN 0022-4049. doi:10.1016/j.jpaa.2021.106884.

[5] Bart Jacobs. Semantics of Weakening and Contraction. *Annals of Pure and Applied Logic*, 69(1):73–106, 1994. ISSN 0168-0072. doi:10.1016/0168-0072(94)90020-5.

[6] Tobias Fritz. A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics. *Advances in Mathematics*, 370: 107239, 2020. ISSN 0001-8708. doi:10.1016/j.aim.2020.107239.

[7] David H. Fremlin. Measure Theory. `https://www.essex.ac.uk/maths/people/fremlin/mt.htm`, 2000–2008.

[8] John Dauns. Categorical W*-Tensor Product. *Transactions of the American Mathematical Society*, 166:439–456, 1972. ISSN 00029947. doi:10.2307/1996061.