Unrestricted Stone Duality for Markov Processes

Robert Furber*, Dexter Kozen†, Kim Larsen*, Radu Mardare* and Prakash Panangaden‡
*Aalborg University, Denmark
†Cornell University
‡McGill University, Montreal, Canada

***Abstract*—Stone duality relates logic, in the form of Boolean algebra, to spaces. Stone-type dualities abound in computer science and have been of great use in understanding the relationship between computational models and the languages used to reason about them. Recent work on probabilistic processes has established a Stone-type duality for a restricted class of Markov processes. The dual category was a new notion—Aumann algebras—which are Boolean algebras equipped with countable family of modalities indexed by rational probabilities. In this article we consider an alternative definition of Aumann algebra that leads to dual adjunction for Markov processes that is a duality for many measurable spaces occurring in practice. This extends a duality for measurable spaces due to Sikorski. In particular, we do not require that the probabilistic modalities preserve a distinguished base of clopen sets, nor that morphisms of Markov processes do so. The extra generality allows us to give a perspicuous definition of event bisimulation on Aumann algebras.**

I. Introduction

Dualities in computer science have enjoyed a recent spate of popular interest. Since Plotkin and Smyth's discovery of a Stone-type duality between the predicate-transformer semantics of Dijkstra and state-transformer semantics [1], [2], it has become increasingly apparent that dualities are ubiquitous in computer science, having appeared in automata and formal language theory, automated deduction, programming language semantics and verification, domain theory, and concurrency theory [3]–[14].

Dualities are important because they establish canonical connections between computational models and the languages used to reason about them. A duality gives an exact characterization of the power of a state transition system by showing how the system determines a corresponding logic or algebra in a canonical way, and vice versa. Moreover, algebra homomorphisms correspond directly to structure-preserving maps or bisimulations of the transition system, allowing mathematical arguments to be transferred in both directions.

The original duality of Stone [15] asserts that the category of Boolean algebras and Boolean algebra homomorphisms is contravariantly equivalent to the category of Stone spaces and continuous maps. Jonsson and Tarski [16] extended Stone's result to Boolean algebras with modal operators and Stone spaces with transitions. A recent surge of interest in probabilistic systems, due largely to impetus from the artificial intelligence and machine learning communities, has led to the study of various logics with constructs for reasoning about probabilities of events or expected behaviour. Recent papers on Markovian logic [11], [17], [18] have established completeness and finite model properties for such systems.

In this paper we focus on the duality of a certain class of models of probabilistic computation, namely *Markov transition systems*, with a certain class of Boolean algebras with operators that behave like probabilistic modalities, the *Aumann algebras*. Aumann algebra is the algebraic analogue of Markovian logic; that is, it is to Markovian logic what Boolean algebra is to propositional logic. They were first defined in [19], where a restricted form of the duality was established. The duality was shown to hold only under certain (somewhat artificial) assumptions, to wit:

- The Aumann algebra must be countable.
- The Borel sets of the Markov transition system must be generated by a distinguished countable family of clopen sets, and morphisms must preserve the distinguished clopens.

These assumptions were made in order to apply the *Rasiowa–Sikorski lemma* [20], a lemma of logic that is dual to the *Baire category theorem* of topology. The RSL/BCT implies that certain "bad" ultrafilters (those not satisfying the countably many infinitary defining conditions of countable Aumann algebras) can be deleted from the Stone space without changing the supported algebra of measurable sets, since the "good" ultrafilters are topologically dense.

Although not a perfect duality due to these restrictions, the groundwork laid in that paper nevertheless led to significant advances in the completeness of Markovian logics [21], [22]. Previously, strong completeness theorems had used a powerful infinitary axiom scheme called the *countable additivity rule*, which has uncountably many instances. Moreover, one needs to postulate *Lindenbaum's lemma* (every consistent set of formulas extends to a maximally consistent set), which for these logics is conjectured but not proven. The duality result of [19] gives rise to a complete axiomatization that does not involve infinitary axiom schemes with uncountably many instances and that satisfies Lindenbaum's lemma.

In this paper we improve the duality of [19] to a full-fledged Stone-like duality between Markov transition systems and Aumann algebras based on Sikorski's Stone duality for σ-perfect σ-fields [23], [24] and σ-spatial Boolean algebras. A σ-Boolean algebra is *σ-spatial* if every element, other than $\perp$, is contained in an σ-complete ultrafilter; this is the algebraic analogue of Lindenbaum's lemma. The construction does not use the RSL/BCT, thus the restrictions mentioned above are no longer necessary. However, we need to change the definition of an Aumann algebra, so this duality is not strictly a generalization of [19]. In particular, the axiom AA8 that we use has uncountably many instances. We say that the duality is Stone-*like* because the duality is no longer an algebraic/topological duality in the strict sense of the word, as the "topological" side is axiomatized in terms of measure-theoretic properties of the state space, not topological properties as with traditional Stone-type results. This is an easy price to pay, because it brings the relevant properties of the state space needed for duality into sharp relief.

Our key results are:

- a version of the duality of [19] for all σ-spatial Aumann algebras, not just countable ones; and
- the removal of the assumptions that Markov processes be countably generated and that maps preserve the distinguished clopens.

The paper is organized as follows. In §II, we briefly review the necessary background material. In §III, we describe Sikorski's Stone duality for measurable spaces. We do this so as to fix notation and also because it will be used in two different places later in the article. It takes the form of an adjunction that becomes a categorical duality when restricted to objects for which the unit and counit are isomorphisms. In §IV, we recall Halmos's description of free σ-Boolean algebras and what a presentation of a σ-Boolean algebra is. We then give a presentation of the Borel σ-field of $[0,1]$ as an abstract σ-Boolean algebra. In §V, we describe an alternative definition of Aumann algebras and prove a duality for Aumann algebras and (discrete-time, continuous-space, time-homogeneous) Markov processes extending the duality for σ-Boolean algebras and measurable spaces in §III. Finally, in §VI we describe the generalization of labelled Markov processes and how event bisimulation is formulated in the setting of Aumann algebras.

II. Background

See Johnstone [25] for a detailed introduction to Stone duality and its ramifications, as well as an account of several other related mathematical dualities such as Priestley duality for distributed lattices and Gelfand duality for C^*-algebras.

We assume knowledge of basic notions from measure theory and topology such as field of sets, σ-field, measurable set, measurable space, measurable function, measure, topology, open and closed sets, continuous functions, and the Borel algebra of a topology (denoted by $\mathcal{B}o(X)$ for X a topological space in this article). See [26], [27] for a more thorough introduction.

Throughout, we use *σ-Boolean algebra* to refer to a σ-complete Boolean algebra, *i.e.* a Boolean algebra with countable joins and meets, and use σ-field to refer to a pair $(X, \mathcal{F})$ where X is a set, and $\mathcal{F}$ a family of subsets that is a σ-Boolean algebra under set-theoretic operations. To avoid confusion we do not use the term σ-algebra.

The natural notion of an ultrafilter on a σ-Boolean algebra is a *σ-ultrafilter*, *i.e.* an ultrafilter that is additionally closed under countable meets. We say that a σ-Boolean algebra is *σ-spatial* if every element is contained in a σ-ultrafilter.[1] If we use σ-homomorphisms as morphisms, σ-Boolean algebras form a category σ-$\mathbf{BA}$, with σ-spatial algebras forming a full subcategory σ-$\mathbf{BA}_{\mathrm{sp}}$.

Define $\mathbb{Q}_0 = \mathbb{Q} \cap [0,1]$ and $\mathbb{R}^+ = \mathbb{R} \cap [0,\infty)$.

A. Measurable Spaces and Measures

If $\mathcal{F} \subseteq \mathcal{P}M$, the *$\sigma$-field generated by $\mathcal{F}$* is the smallest σ-field containing $\mathcal{F}$.

In any measurable space (X, Σ), a point $x \in X$ defines a σ-ultrafilter on Σ by

$$\langle x \rangle = \{S \in \Sigma \mid x \in S\}.$$

Sikorski introduced the term *σ-perfect* for those measurable spaces for which $\langle\text{-}\rangle$ is a bijection from X to the set of σ-ultrafilters on Σ [24, pp. 1, 98].

Measurable spaces form a category $\mathcal{M}es$ with measurable maps as morphisms, and σ-perfect measurable spaces form a full subcategory $\mathcal{PM}es$.

A nonnegative real-valued set function μ is said to be *finitely additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$, and *countably additive* if $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ for a countable pairwise-disjoint family of measurable sets. A *measure* on a measurable space (X, Σ) is a countably additive set function $\mu : \Sigma \to \mathbb{R}^+$. A measure is a *probability measure* if in addition $\mu(X) = 1$, and a *subprobability measure* if $\mu(X) \leq 1$. We use $\mathcal{G}(X, \Sigma)$ to denote the set of subprobability measures on (X, Σ).

We can view $\mathcal{G}(X, \Sigma)$ as a measurable space by considering the σ-field generated by the sets $\{\mu \in \mathcal{G}(X, \Sigma) \mid \mu(S) \geq r\}$ for $S \in \Sigma$ and $r \in [0,1]$. This is the least σ-field on $\mathcal{G}(X, \Sigma)$ such that all maps $\mu \mapsto \mu(S) : \mathcal{G}(X, \Sigma) \to [0,1]$ for $S \in \Sigma$ are measurable, where the real interval $[0,1]$ is endowed its Borel σ-field. In the usual way, we extend $\mathcal{G}$ to a functor $\mathcal{M}es \to \mathcal{M}es$ by defining

$$\mathcal{G}(f)(\mu)(T) = \mu(f^{-1}(T)),$$

where $f : (X, \Sigma) \to (Y, \Theta)$ is a measurable map, $\mu \in \mathcal{G}(X, \Sigma)$ and $T \in \Theta$. It is worth mentioning that $\mathcal{G}$ is the subprobabilistic Giry monad [28].

B. Markov Processes

Markov processes (MPs) are models of probabilistic systems with a continuous state space and discrete-time probabilistic transitions [28]–[30].

Definition 1 (Markov process)**.** *A* Markov process *(MP) is a measurable space (X, Σ) equipped with a measurable map $\theta : (X, \Sigma) \to \mathcal{G}(X, \Sigma)$. A Markov process is said to be σ-perfect iff (X, Σ) is. Maps of Markov processes are "zig-zags",* i.e. *if $(X_1, \Sigma_1, \theta_1)$ and $(X_2, \Sigma_2, \theta_2)$ are Markov*

[1] This terminology is inspired by the theory of locales.

processes, a measurable map $f : (X_1, \Sigma_1) \to (X_2, \Sigma_2)$ is a map of Markov processes if

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "\theta_1"'] & X_2 \arrow[d, "\theta_2"] \\ \mathcal{G}(X_1) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(X_2) \end{tikzcd}$$

commutes. Then Markov processes and morphisms thereof form a category $\mathbf{Markov}$*, with σ-perfect Markov processes forming a full subcategory* $\mathbf{PMarkov}$.

In a Markov process (X, Σ, θ), and θ is called the *transition function*. For $x \in X$, $\theta(x) : \Sigma \to [0, 1]$ is a subprobability measure on the state space (X, Σ). For $S \in \Sigma$, the value $\theta(x)(S) \in [0, 1]$ represents the probability of a transition from x to a state in S.

The condition that θ be a measurable function $X \to \mathcal{G}(X, \Sigma)$ is equivalent to the condition that for fixed $S \in \Sigma$, the function $x \mapsto \theta(x)(S)$ is a measurable function $X \to [0, 1]$ (see e.g. [30, Proposition 2.9]).

C. Aumann Algebras

Aumann Algebra (AA) [19] is the algebraic analogue of Markovian logic [11], [17], [18]. It is so named in honor of Robert Aumann, who has made fundamental contributions to probabilistic logic [31].

Definition 2 (Aumann algebra)**.** *A σ-Aumann algebra is a tuple $(A, (L_r)_{r \in \mathbb{Q}_0})$, where A is a σ-Boolean algebra, $\mathbb{Q}_0 = \mathbb{Q} \cap [0, 1]$ and each $L_r : A \to A$, such that the following axioms hold, where a, b are arbitrary elements of A and r, s are elements of $\mathbb{Q}_0$:*

(AA1) $\top \leq L_0(a)$

(AA2) $L_r(\bot) \leq \bot$, where $r > 0$

(AA3) $L_r(a) \leq \neg L_s(\neg a)$ if $r + s > 1$

(AA4) $L_r(a \wedge b) \wedge L_s(a \wedge \neg b) \leq L_{r+s}(a)$ if $r + s \leq 1$

(AA5) $\neg L_r(a \wedge b) \wedge \neg L_s(a \wedge \neg b) \leq \neg L_{r+s}(a)$ if $r + s \leq 1$

(AA6) $a \leq b$ implies $L_r(a) \leq L_r(b)$

(AA7) $\bigwedge_{r<s} L_r(a) = L_s(a)$

(AA8) If $r \in \mathbb{Q}_0$ and $r \neq 0$, for all countable descending chains $a_1 \geq a_2 \geq \cdots$ such that $\bigwedge_{i=1}^{\infty} a_i = \bot$ we have $\bigwedge_{i=1}^{\infty} L_r(a_i) = \bot$.

TABLE I
AUMANN ALGEBRA

We define a morphism of σ-Aumann algebras $f : (A, (L_r)) \to (B, (M_r))$ to be a σ-Boolean homomorphism such that $f(L_r(a)) = M_r(f(a))$ for all $a \in A$, i.e. *such that the following diagram commutes for all $r \in \mathbb{Q}_0$:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "L_r"'] & B \arrow[d, "M_r"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

We define the category $\mathbf{Aumann}$ *to have σ-Aumann algebras as objects and σ-Aumann algebra morphisms as its morphisms, and* $\mathbf{AumannSp}$ *to be the full subcategory on σ-spatial Aumann algebras.*

The reader may verify that AA1 and AA3-AA6 are the same as in [19, §4.1], and that AA7 and AA8 imply the AA7 of the original definition. Note that AA2 originates in [22, Table 3], and is the small change needed to account for subprobability distributions. Note also that AA2 with $r = 0$ is inconsistent with AA1. Additionally, AA1 is implied by AA7 with $s = 0$. A σ-spatial Aumann algebra is defined to be a σ-Aumann algebra whose underlying σ-Boolean algebra is σ-spatial.

III. Duality for Measurable Spaces

In this section, we express Sikorski's duality for measurable spaces [24, §24, §32][23, 2.1-2] as an adjunction, and describe how this adjunction can be restricted to an equivalence (see [32, Part 0, Proposition 4.2] for a proof that this is possible for any adjunction).

If (X, Σ) is a measurable space, then Σ is a σ-Boolean algebra. We can use this to define a functor $F : \mathcal{M}es \to \sigma\text{-}\mathbf{BA}^{\mathrm{op}}$ by defining $F(X, \Sigma) = \Sigma$, and for $f : (X, \Sigma) \to (Y, \Theta)$ a measurable map and $T \in \Theta$

$$F(f)(T) = f^{-1}(T).$$

This is easily verified to be a functor. The following is easy to verify using the fact that sets can be distinguished by their points.

Lemma 3. *For any measurable space (X, Σ), Σ is a σ-spatial σ-Boolean algebra.*

So we can also regard F as having the type $\mathcal{M}es \to \sigma\text{-}\mathbf{BA}^{\mathrm{op}}_{\mathrm{sp}}$. The analogous functor in Stone duality takes the Boolean algebra of clopens of a Stone space.

The reader familiar with Stone duality will already be expecting ultrafilters to be involved in the definition of a functor the other way. For A a σ-Boolean algebra, we define $\mathcal{U}^\sigma(A)$ to be the set of all σ-ultrafilters on A. Given an element $a \in A$, we define

$$(\!|a|\!) = \{u \in \mathcal{U}^\sigma(A) | u \ni a\} \subseteq \mathcal{U}^\sigma(A).$$

We can define $\mathcal{F}(A) \subseteq \mathcal{P}(\mathcal{U}^\sigma(A))$ by $\mathcal{F}(A) = (\!|A|\!)$, *i.e.* it is the image of A under the map $(\!|\text{-}|\!)$.

Lemma 4. $(\!|\text{-}|\!)$ *is a surjective morphism of σ-Boolean algebras $A \to \mathcal{F}(A)$, and $\mathcal{F}(A)$ is a σ-field [24, §24.1]. We also have that A is σ-spatial iff $(\!|\text{-}|\!)$ is an isomorphism.*

We give the proof of this, and the other results whose proof is omitted in the main text, in the appendix.

We can now define $G : \sigma\text{-}\mathbf{BA}^{\mathrm{op}} \to \mathcal{PM}es$ on objects as $G(A) = (\mathcal{U}^\sigma(A), \mathcal{F}(A))$, where $\mathcal{F}(A) = (\!|A|\!)$. On σ-homomorphisms $f : A \to B$, $G(f)$ is defined for each $u \in \mathcal{U}^\sigma(B)$ as

$$G(f)(u) = f^{-1}(u).$$

In order to prove that $F \dashv G$, we define the unit and counit of the adjunction. For a measurable space (X, Σ), for each element $x \in X$, we can define an ultrafilter $\langle x \rangle \in \mathcal{U}^\sigma(\Sigma)$ as

$$\langle x \rangle = \{S \in \Sigma \mid x \in S\}.$$

We define the unit $\eta_X : (X, \Sigma) \to G(F(X, \Sigma))$ and counit $\epsilon_A : F(G(A)) \to A$ to be

$$\eta_X(x) = \langle x \rangle$$
$$\epsilon_A(a) = (\!| a |\!).$$

The direction of ϵ_A is reversed because we use $\sigma\text{-}\mathbf{BA}^{\mathrm{op}}$.

Theorem 5. *(F, G, η, ϵ) is an adjunction making G a right adjoint to F. In fact, G maps into $\mathcal{PM}es$ and by restricting F and G to the categories where the unit and counit are isomorphisms, they define an adjoint equivalence $\sigma\text{-}\mathbf{BA}_{\mathrm{sp}}^{\mathrm{op}} \simeq \mathcal{PM}es$.*

In passing, the above theorem shows that $\mathcal{PM}es$ and $\sigma\text{-}\mathbf{BA}_{\mathrm{sp}}$ are *reflective* [33, §IV.3] subcategories of $\mathcal{M}es$ and $\sigma\text{-}\mathbf{BA}$, respectively.

The reader might object to the definition of a σ-perfect measurable space as only attempting to "solve" a problem by defining it out of existence. To address this potential criticism, we show that there are many σ-perfect measurable spaces occurring in practice using a theorem of Hewitt. First, we recall that on a topological space X, the Baire σ-field $\mathcal{B}a(X)$ can be defined to be the σ-field generated by the *zero sets*, the subsets of X of the form $f^{-1}(0)$ for some continuous map $X \to \mathbb{R}$. If X is metrizable, $\mathcal{B}a(X)$ is the same as the Borel σ-field. We also need to refer to the concept of a realcompact space. We omit the definition [34, §5.9], but we only need the fact that every σ-compact Hausdorff space and every separably metrizable space[2] is realcompact, as is shown in [34, §8.2].

Theorem 6 (Hewitt)**.** *For a completely regular space X, $(X, \mathcal{B}a(X))$ is σ-perfect iff X is realcompact.*

See [35, Theorem 16] for the proof[3].

Therefore the Borel σ-field of any separable metric space, *e.g.* a Polish space or analytic space, and the Baire σ-field of any compact Hausdorff space are σ-perfect. We warn the reader that it is not the case that the Baire σ-field of a *locally* compact space is σ-perfect, nor the Borel σ-field of an unmetrizable compact Hausdorff space, and σ-perfectness is not preserved under σ-subfields (even though σ-spatiality is preserved under subalgebras).

For example, consider an uncountable set X. Take Σ to consist of countable sets and their complements (co-countable sets), the countable co-countable σ-field. Then we define u to be the set of co-countable sets. This is a non-principal σ-ultrafilter on Σ, so (X, Σ) is a measurable space that is not σ-perfect. But note that if we take $X = \mathbb{R}$, this is a σ-subfield of the Borel σ-field.

One thing to note is that the smallest cardinality of a set X such that $(X, \mathcal{P}(X))$ is *not* σ-spatial is strongly inaccessible, *i.e.* it is not possible to produce this set by taking powersets or unions of strictly smaller families of strictly smaller sets. This was first proven by Ulam [36, Lemma 1 and Satz 4]. It is therefore not possible to prove the existence, or even the consistency, of a non-σ-spatial discrete set [37, Theorem 12.12].

[2]Not requiring completeness.

[3]Hewitt uses *Q*-space to mean realcompact space.

IV. Presentations of σ-Boolean Algebras

In this section we describe Halmos's construction of the free σ-Boolean algebra on a set, how to give presentations of σ-fields in terms of generators and relations, how to define measurable maps in terms of presentations, and give a presentation of $\mathcal{B}o([0,1])$. We note that this point that presentations can be defined in a more general setting of universal algebra for theories with operations of countable arity, and Lemma 8 and Proposition 9 (i) could have been given as a reference to such a general theorem rather than being proved in this special setting. However, we will stick with a presentation closer to measure theory than universal algebra.

We have the usual forgetful functor $U : \sigma\text{-}\mathbf{BA} \to \mathbf{Set}$.

Proposition 7 (Halmos)**.** *The functor U has a left adjoint H, given on objects by*

$$H(X) = \mathcal{B}a(2^X),$$

where 2^X is given the product topology. This is also a left adjoint to the restriction of U to $\sigma\text{-}\mathbf{BA}_{\mathrm{sp}}$.

We give the extra parts of the proof needed to get from [38, §23, Theorem 14] to the above statement in the appendix.

We have already seen the definition of a σ-ultrafilter. A σ-*ideal* in a σ-Boolean algebra A is a subset $I \subseteq A$ that is downward closed (*i.e.* if $b \leq a$ and $a \in I$, then $b \in I$) and closed under countable joins. As any σ-ideal is an ideal, we can define A/I to be the set of equivalence classes of elements of A modulo the relation $a \sim b \Leftrightarrow a \triangle b \in I$, where $\triangle$ is the symmetric difference, as usual, and σ-Boolean operations are well-defined with respect to this equivalence relation.

If K is a subset of a σ-Boolean algebra A, we can define the sub-σ-Boolean algebra B *generated* by K to be the smallest σ-Boolean algebra in A containing K. This can equivalently be defined as either the intersection of all σ-Boolean subalgebras of A that contain K, or by building up elements of B as σ-Boolean combinations of elements of K using countable ordinals, as in the Borel hierarchy [39, §II.3]. If $B = A$, we say that A is *generated* by K. Note that K will not necessarily generate A as a Boolean algebra, in general countable operations will be necessary.

Recall that $H : \mathbf{Set} \to \sigma\text{-}\mathbf{BA}$ is the free (σ-spatial) σ-Boolean algebra on a set, from Proposition 7, in the following lemma.

Lemma 8. *A set $K \subseteq A$ generates A iff the universal map $\tilde{i} : H(K) \to A$:*

$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[dr, "i"'] & U(H(K)) \arrow[d, "U\tilde{i}"] & H(K) \arrow[d, dotted, "\tilde{i}"] \\ & U(A) & A \end{tikzcd}$$

is surjective, where i is the inclusion morphism.

We say a σ-ideal $I \subseteq A$ is *generated* by a subset $R \subseteq I$ if I is the smallest σ-ideal containing R, equivalently if

$$I = \left\{ a \in A \mid \exists (b_i)_{i \in \mathcal{N}} . \forall i \in \mathcal{N} . b_i \in R \text{ and } a \leq \bigvee_{i=1}^{\infty} b_i \right\} .$$

Note that R does not necessarily generate I as an ideal, as countable joins may be necessary to produce every element of I. Also note that the σ-ideal generated by a set and the σ-Boolean algebra generated by a set are not necessarily the same.

A σ-ideal is *principal* if it is generated by one element, and if $(b_i)_{i \in \mathcal{N}}$ is a countable set of generators for a σ-ideal I, then I is generated by $\bigvee_{i \in \mathcal{N}} b_i$, *i.e.* every countably generated σ-ideal is principal.

A *presentation* of a σ-Boolean algebra A is a pair (K, R) where $K \subseteq A$ generates the σ-Boolean algebra A and R generates the σ-ideal $\tilde{i}^{-1}(\bot)$ in $H(K)$, where $i : K \to A$ is the inclusion morphism. We call the elements of R *relations*. This agrees with the usual notion of a presentation of a group or ring in terms of generators and relations. In view of Theorem 5, we can define a presentation of a σ-perfect measurable space (X, Σ) to be a presentation of Σ.

Once we have a presentation of a σ-Boolean algebra, we can define homomormorphisms by giving their values on the generators and checking that the relations are satisfied.

Proposition 9.

(i) Let A and B be σ-Boolean algebras, (K, R) a presentation of A, and $f : K \to B$ a function. There exists a σ-homomorphism $g : A \to B$ such that $g|_K = f$ iff $\tilde{f}(r) = \bot$ for all $r \in R$.

(ii) Let (X, Σ) and (Y, Θ) be measurable spaces, where (X, Σ) is σ-perfect, and let (K, R) be a presentation of (X, Σ). Let $f : K \to \Theta$ be a function. There exists a measurable map $g : (Y, \Theta) \to (X, \Sigma)$ such that $g^{-1}|_K = f$ iff $\tilde{f}(r) = \emptyset$ for all $r \in R$.

In the special case that there are only countably many relations, we can verify that a set of relations is sufficient to define a presentation in another way.

Lemma 10. *Let (X, Σ) be a σ-perfect measurable space, $K \subseteq \Sigma$ a set of generators with inclusion morphism $i : K \to \Sigma$, and $(r_j)_{j \in J}$ a countable set of relations,* i.e. *elements of $H(K)$ such that $\tilde{i}(r_j) = \emptyset$. The following condition implies that $(K, (r_j)_{j \in J})$ is a presentation of (X, Σ): For all $u \in \mathcal{U}^{\sigma}(H(K))$ such that $\forall j \in J . r_j \notin u$, we have that there exists $x \in X$ such that $\tilde{i}^{-1}(\langle x \rangle) = u$.*

We can now give a presentation of $\mathcal{B}o([0,1])$ for later use. This presentation is related to a presentation of $\mathcal{B}o([-\infty, \infty])$ given by Sikorski [23, II.3 Lemma].

Define

$$K = \{[r, 1] \mid r \in \mathbb{Q}_0\} \tag{1}$$

This is a countable family of closed subsets of $[0, 1]$. To define the relations, we write

$$B_r = \eta_K([r, 1])$$

Then we define the relations as

$$R = \{(B_r \wedge B_s) \triangle B_s\}_{r<s} \cup \left\{ \left(\bigwedge_{r<s} B_r \right) \triangle B_s \right\}_{s \in \mathbb{Q}_0} , \tag{2}$$

where r and s are understood to range over $\mathbb{Q}_0$.

Lemma 11. *The above (K, R), as in* (1) *and* (2)*, define a presentation of* $([0,1], \mathcal{B}o([0,1]))$.

V. Duality for Markov Processes

In this section, we extend the adjunction and duality from Section III to Markov processes. Recall that the category of Markov processes is called $\mathbf{Markov}$ and the category of Aumann algebras is $\mathbf{Aumann}$. We define a σ-perfect Markov process to be one whose underlying measurable space is σ-perfect, forming the full subcategory $\mathbf{PMarkov}$. Likewise, an Aumann algebra is called σ-spatial if its underlying σ-Boolean algebra is σ-spatial, and these form a full subcategory $\mathbf{AumannSp}$.

As σ-perfectness of a Markov process only depends on the underlying measurable space, our remarks at the end of Section III imply that the usual Markov processes defined on the Borel σ-fields of Polish spaces or analytic spaces are σ-perfect, so the duality works as a categorical equivalence in these cases.

When defining the adjunction, it is useful to recall Giry's definition of p_S, where (X, Σ) is a measurable space and $S \in \Sigma$

$$p_S : \mathcal{G}(X, \Sigma) \to [0,1]$$
$$p_S(\nu) = \nu(S).$$

The σ-field of $\mathcal{G}(X, \Sigma)$ is defined to be the coarsest such that p_S is measurable, equivalently that generated by $p_S^{-1}(B)$ as S varies over all $S \in \Sigma$ and B varies over the Borel sets of $[0, 1]$, or equivalently any family of sets generating $\mathcal{B}o([0,1])$.

As in the case of $\mathcal{M}es$ and σ-$\mathbf{BA}$, we define $F : \mathbf{Markov} \to \mathbf{AumannSp}^{\mathrm{op}}$ based on $F : \mathcal{M}es \to \sigma\text{-}\mathbf{BA}_{\mathrm{sp}}^{\mathrm{op}}$ and $G : \mathbf{Aumann}^{\mathrm{op}} \to \mathbf{PMarkov}$ based on $G : \sigma\text{-}\mathbf{BA}^{\mathrm{op}} \to \mathcal{PM}es$.

For a Markov process (X, Σ, θ), and a morphism of Markov processes $f : (X, \Sigma, \theta) \to (Y, \Theta, \lambda)$ we define F as

$$F(X, \Sigma, \theta) = (\Sigma, (L_r)_{r \in \mathbb{Q}_0})$$
$$F(f) = f^{-1},$$

where L_r is defined, for $S \in \Sigma$, as

$$L_r(S) = \{x \in X \mid \theta(x)(S) \geq r\} \tag{3}$$

Proposition 12. *F defines a a functor* $\mathbf{Markov} \to \mathbf{AumannSp}^{\mathrm{op}}$.

We can now define the Markov process arising from a σ-Aumann algebra, defining the functor $G : \mathbf{Aumann}^{\mathrm{op}} \to \mathbf{PMarkov}$ on objects.

Given an Aumann algebra $(A, (L_r)_{r\in\mathbb{Q}_0})$ we define, using Proposition 9, for each $a \in A$ a measurable map $\theta_a : \mathcal{U}^\sigma(A) \to [0,1]$ such that

$$\theta_a^{-1}([r,1]) = (\!|L_r(a)|\!), \tag{4}$$

and then define $\theta : \mathcal{U}^\sigma(A) \to \mathcal{G}(\mathcal{U}^\sigma(A))$ as

$$\theta(u)((\!|a|\!)) = \theta_a(u). \tag{5}$$

Proposition 13. *If $(A, (L_r)_{r\in\mathbb{Q}_0})$ is a σ-Aumann algebra, $(\mathcal{U}^\sigma(A), \mathcal{F}(A), \theta)$ is a σ-perfect Markov process.*

We can now show that this defines a functor $G : \mathbf{Aumann}^{\mathrm{op}} \to \mathbf{PMarkov}$. On objects, we should have

$$G(A, (L_r)_{r\in\mathbb{Q}_0}) = (\mathcal{U}^\sigma(A), \mathcal{F}(A), \theta),$$

as described above. Given a map of σ-Aumann algebras $g : (A, (L_r)_{r\in\mathbb{Q}_0}) \to (B, (M_r)_{r\in\mathbb{Q}_0})$ we define $G(g)$ exactly as for σ-Boolean algebras, *i.e.* $G(g)(u) = g^{-1}(u)$.

Proposition 14. *With the above definition, G is a functor* $\mathbf{Aumann}^{\mathrm{op}} \to \mathbf{PMarkov}$.

Theorem 15. *F is a left adjoint to G, and when restricted they define adjoint equivalences* $\mathbf{AumannSp}^{\mathrm{op}} \simeq \mathbf{PMarkov}$.

Note that by Theorem 6 this duality can be applied to any of the Stone-Markov processes with countable base considered in [19], although the dual Aumann algebra will be the Borel sets, not the base of clopens.

VI. Event Bisimulation and Duality for Labelled Markov Processes

Given a measurable space (X, Σ), a *labelled Markov process* is a tuple $(X, \Sigma, (\theta^e)_{e\in E})$, where E is a set of *labels* and for each $e \in E$, $\theta^e : X \to \mathcal{G}(X, \Sigma)$ is a measurable function. If $(X, \Sigma, (\theta^e)_{e\in E})$ and $(Y, \Theta, (\lambda^e)_{e\in E})$ are labelled Markov processes with the same label set E, we say that a measurable function $f : (X, \Sigma) \to (Y, \Theta)$ is a *morphism of labelled Markov processes* if it is a morphism of Markov processes $f : (X, \Sigma, \theta_e) \to (Y, \Theta, \lambda_e)$ for each $e \in E$. For any set of labels, we have a category $\mathbf{LabMarkov}_E$ of E-labelled Markov processes and their morphisms. It should now be obvious how to define the full subcategory of σ-perfect labelled Markov processes, $\mathbf{PLabMarkov}_E$.

We can define a *labelled σ-Aumann algebra* to be $(A, (L_r^e)_{e\in E, r\in\mathbb{Q}_0})$, such that for each label $e \in E$, $(A, (L_r^e)_{r\in\mathbb{Q}_0})$ is a σ-Aumann algebra. A morphism of E-labelled σ-Aumann algebras $(A, (L_r^e)) \to (B, (M_r^e))$ is a σ-Boolean homomorphism $A \to B$ that is a σ-Aumann algebra homomorphism for each $e \in E$. For each set of labels E, we have categories $\mathbf{LabAumann}_E$ and $\mathbf{LabAumannSp}_E$ defined in the familiar way. By working with each $e \in E$ independently, we can define F, G and a duality as in Theorem 15.

In the context of labelled Markov processes, an *event bisimulation* on $(X, \Sigma, (\theta^e)_{e\in E})$ is defined to be a sub σ-field $\Lambda \subseteq \Sigma$ such that $(X, \Lambda, (\theta^e|_\Lambda)_{e\in E})$ is a labelled Markov process, where for each $e \in E$, $\theta^e|_\Lambda : X \to \mathcal{G}(X, \Lambda)$ is the function such that for each $x \in X$, $\theta^e|_\Lambda(x)$ is the restriction of $\theta^e(x)$ to Λ. This notion was originally defined in [40, Definition 4.3], as a version of the notion of probabilistic bisimulation [41] that is more adapted to probabilistic logics.

Theorem 16. *Let $(X, \Sigma, (\theta^e)_{e\in E})$ be a labelled Markov process. A σ-field $\Lambda \subseteq \Sigma$ is an event bisimulation iff it is a σ-Aumann subalgebra of $F(X, \Sigma, (\theta^e)_{e\in E})$,* i.e. *iff it is preserved by the Aumann algebra operations.*

Proof. We start with the only if direction, which is to say we show that an event bisimulation Λ is also an Aumann subalgebra of the Aumann algebra $F(X, \Sigma, (\theta^e))$.

Since an event bisimulation is a σ-field supporting a labelled Markov process, when F is applied to it becomes an Aumann algebra. As $\theta^e|_\Lambda$ is the restriction of θ^e to Λ at all points $x \in X$, we get that Λ and Σ agree on the effect of the L_r^e operators for all $r \in \mathbb{Q}_0$. This shows that Λ is an Aumann subalgebra of Σ.

For the other direction, suppose A_0 is an Aumann subalgebra of $F(X, \Sigma, (\theta^e)_{e\in E})$. We prove that $\mathcal{A}_0$ is an event bisimulation of $(X, \Sigma, (\theta^e)_{e\in E})$ as follows. We need to show that for each $e \in E$,

$$\theta^e|_{\mathcal{A}_0} : (X, \mathcal{A}_0) \to \mathcal{G}(X, \mathcal{A}_0)$$

is measurable. This is equivalent to showing that for each $a \in \mathcal{A}_0$ and each $B \in \mathcal{B}o([0,1])$,

$$(\theta^e|_{\mathcal{A}_0})^{-1}(p_a^{-1}(B)) \in A_0.$$

To do this, it is sufficient to prove that for any rational $r \leq 1$,

$$(\theta^e|_{\mathcal{A}_0})^{-1}(p_a^{-1}([r,1])) \in A_0.$$

But since $a \in \mathcal{A}_0$, we get

$$(\theta^e|_{\mathcal{A}_0})^{-1}(p_a^{-1}([r,1])) = L_r^e a \in \mathcal{A}_0$$

and this concludes the proof. □

Therefore we can, if we like, define event bisimulations directly on σ-Aumann algebras, by taking them to be σ-Aumann subalgebras.

VII. Conclusion

We have given a general Stone-like duality between spatial Aumann algebras and certain Markov processes, improving a similar duality of [19] of a more restricted form. We have also shown how the improved version captures the notion of event bisimulation for Markov processes.

Strictly speaking, the result of [19] is not a special case of the result of this paper, because we have amended the definition of Aumann algebras to assume countable completeness:

all countable joins are assumed to exist, not just the definable ones. This result is probably not the last word on the subject, as it may be possible to derive an even more general version of the duality parameterized by the class of joins that are assumed to exist that would subsume both the results of [19] and those of this paper. We leave such investigations to future work.

ACKNOWLEDGEMENTS

Robert Furber is supported by the Danish Council for Independent Research, Project 4181-00360. Dexter Kozen is supported by the National Science Foundation, grant CCF1637532, and by the National Security Agency.

REFERENCES

[1] G. D. Plotkin, "Lecture Notes on Domain Theory," 1983, available from his home page as The Pisa Notes.
[2] M. Smyth, "Powerdomains and Predicate Transformers," in *ICALP*, J. Diaz, Ed. Springer-Verlag, 1983, pp. 662–676, lecture Notes In Computer Science 154.
[3] D. Kozen, "A probabilistic PDL," *Journal of Computer and Systems Sciences*, vol. 30, no. 2, pp. 162–178, 1985.
[4] S. Abramsky, "Domain Theory in Logical Form," *Annals of Pure and Applied Logic*, vol. 51, pp. 1–77, 1991.
[5] M. M. Bonsangue and A. Kurz, "Duality for Logics of Transition Systems," in *FoSSaCS*, 2005, pp. 455–469.
[6] F. Bonchi, M. Bonsangue, J. Rutten, and A. Silva, "Brzozowski's algorithm (co)algebraically," in *Logics and Program Semantics: Essays Dedicated to Dexter Kozen*, ser. Lecture Notes In Computer Science, vol. 7230. Springer-Verlag, 2012, pp. 12–23.
[7] N. Bezhanishvili, C. Kupke, and P. Panangaden, "Minimization via Duality," in *WoLLIC 2012 Proceedings*, ser. Lecture Notes in Computer Science, vol. 7456. Springer, 2012, pp. 191–205.
[8] M. Gehrke, S. Grigorieff, and J.-E. Pin, "Duality and Equational Theory of Regular Languages," in *ICALP (2)*, 2008, pp. 246–257.
[9] B. Jacobs, "Probabilities, distribution monads, and convex categories," *Theor. Comput. Sci.*, vol. 412, no. 28, pp. 3323–3336, 2011.
[10] M. Mislove, J. Ouaknine, D. Pavlovic, and J. Worrell, "Duality for Labelled Markov Processes," in *FOSSACS*, ser. Lecture Notes In Computer Science, I. Walukiewicz, Ed., vol. 2987, 2004, pp. 393–407.
[11] R. Goldblatt, "On the role of the Baire category theorem in the foundations of logic," *Journal of Symbolic logic*, pp. 412–422, 1985.
[12] ——, "Deduction systems for coalgebras over measurable spaces," *Journal of Logic and Computation*, vol. 20, no. 5, pp. 1069–1100, 2010.
[13] M. J. Gabbay, T. Litak, and D. Petrisan, "Stone duality for nominal Boolean algebras with 'new'," in *4th Int. Conf. Algebra and Coalgebra in Computer Science (CALCO 2011)*, ser. LNCS, vol. 6859. Springer, 2011, pp. 192–207.
[14] M. A. Moshier and D. Petrisan, "A duality theorem for real C*-algebras," in *3rd Int. Conf. Algebra and Coalgebra in Computer Science (CALCO 2009)*, ser. LNCS, vol. 5728. Springer, 2009, pp. 284–299.
[15] M. H. Stone, "The theory of representations for boolean algebras," *Trans. Amer. Math. Soc.*, vol. 40, pp. 37–111, 1936.
[16] B. Jónsson and A. Tarski, "Boolean algebras with operators I," *American Journal of Mathematics*, vol. 73, pp. 891–939, 1951.
[17] L. Cardelli, K. G. Larsen, and R. Mardare, "Continuous Markovian Logic - From Complete Axiomatization to the Metric Space of Formulas," in *CSL*, 2011, pp. 144–158.
[18] C. Zhou, "A complete deductive system for probability logic with application to Harsanyi type spaces," Ph.D. dissertation, Indiana University, 2007.
[19] D. Kozen, K. G. Larsen, R. Mardare, and P. Panangaden, "Stone Duality for Markov Processes," in *Proceedings of the Twenty-Eighth Annual IEEE Symposium on Logic in Computer Science (LICS 2013)*. IEEE Computer Society Press, June 2013, pp. 321–330.
[20] H. Rasiowa and R. Sikorski, "A proof of the completeness theorem of Gödel," *Fund. Math*, vol. 37, pp. 193–200, 1950.
[21] R. Goldblatt, "The Countable Henkin Principle," in *The Life and Work of Leon Henkin: Essays on His Contributions*, M. Manzano, I. Sain, and E. Alonso, Eds. Cham: Springer International Publishing, 2014, pp. 179–201. [Online]. Available: http://dx.doi.org/10.1007/978-3-319-09719-0_13
[22] D. Kozen, R. Mardare, and P. Panangaden, "Strong Completeness for Markovian Logics," in *Mathematical Foundations of Computer Science 2013: 38th International Symposium, MFCS 2013, Klosterneuburg, Austria, August 26-30, 2013. Proceedings*, K. Chatterjee and J. Sgall, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 655–666.
[23] R. Sikorski, "On the Inducing of Homomorphisms by Mappings," *Fundamenta Mathematicae*, vol. 36, pp. 7–22, 1949.
[24] ——, *Boolean Algebras*, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1969, no. 25.
[25] P. Johnstone, *Stone Spaces*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982, vol. 3.
[26] P. Billingsley, *Probability and Measure*. Wiley-Interscience, 1995.
[27] R. M. Dudley, *Real Analysis and Probability*. Wadsworth and Brookes/Cole, 1989.
[28] P. Panangaden, *Labelled Markov Processes*. Imperial College Press, 2009.
[29] J. Desharnais, A. Edalat, and P. Panangaden, "Bisimulation for Labeled Markov Processes," *Information and Computation*, vol. 179, no. 2, pp. 163–193, Dec 2002.
[30] E.-E. Doberkat, *Stochastic Relations. Foundations for Markov Transition Systems*. New York: Chapman and Hall, 2007.
[31] R. Aumann, "Interactive epistemology I: knowledge, II probability," *International Journal of Game Theory*, vol. 28, pp. 263–314, 1999.
[32] J. Lambek and P. Scott, *Introduction to Higher Order Categorical Logic*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1986, vol. 7.
[33] S. Mac Lane, *Categories for the Working Mathematician*, ser. Graduate Texts in Mathematics. Springer Verlag, 1971.
[34] L. Gillman and M. Jerison, *Rings of Continuous Functions*, ser. Graduate Texts in Mathematics. Springer Verlag, 1976.
[35] E. Hewitt, "Linear Functionals on Spaces of Continuous Functions," *Fundamenta Mathematicae*, vol. 37, pp. 161–189, 1950.
[36] S. Ulam, "Zur Masstheorie in der allgemeinen Mengenlehre," *Fundamenta Mathematicae*, vol. 16, pp. 140–150, 1930.
[37] T. Jech, *Set Theory*, ser. Springer Monographs in Mathematics. Springer, 2003.
[38] P. R. Halmos, *Lectures on Boolean Algebras*, ser. Van Nostrand Mathematical Studies. D. van Nostrand Company, Inc., 1963.
[39] J. Słomiński, "The Theory of Abstract Algebras with Infinitary Operations," *Rozprawy Matematyczne*, vol. 18, pp. 1–67, 1959.
[40] V. Danos, J. Desharnais, F. Laviolette, and P. Panangaden, "Bisimulation and cocongruence for probabilistic systems," *Information and Computation*, vol. 204, no. 4, pp. 503–523, 2006.
[41] K. G. Larsen and A. Skou, "Bisimulation through Probabilistic Testing," *Information and Computation*, vol. 94, pp. 1–28, 1991.
[42] P. R. Halmos, *Measure Theory*, ser. Graduate Texts in Mathematics. Springer-Verlag, 1974, no. 18, originally published in 1950.

APPENDIX

Lemma 4. *$(\!|\text{-}|\!)$ is a surjective morphism of σ-Boolean algebras $A \to \mathcal{F}(A)$, and $\mathcal{F}(A)$ is a σ-field [24, §24.1]. We also have that A is σ-spatial iff $(\!|\text{-}|\!)$ is an isomorphism.*

Proof. By definition, $(\!|\text{-}|\!)$ is surjective from A to $\mathcal{F}(A)$. We first show that $(\!|\text{-}|\!)$ is a σ-Boolean homomorphism, with respect to the set theoretic Boolean operations on $\mathcal{F}(A)$. This implies that $\mathcal{F}(A)$ is a σ-field. We can see that $(\!|\text{-}|\!)$ preserves countable meets by using the fact that σ-filters are closed under countable intersections. The map $(\!|\text{-}|\!)$ preserves complements because if $a \in A$ and $u \in \mathcal{U}^\sigma(A)$, $\neg a \in u$ iff $a \notin u$. So $(\!|\text{-}|\!)$ is a σ-homomorphism and $\mathcal{F}(A)$ is a σ-field.

We now show that $(\!|\text{-}|\!)$ is injective iff $\mathcal{F}(A)$ is σ-spatial. If $a, b \in A$, with $a \neq b$, then at least one of $a \wedge \neg b$ or $b \wedge \neg a$

is different from $\bot$. Without loss of generality take it to be $a \wedge \neg b$. As A is σ-spatial, there is a σ-ultrafilter u containing $a \wedge \neg b$. By up-closedness, this σ-ultrafilter contains a and $\neg b$, so $b \notin u$. Therefore $u \in \llparenthesis a \rrparenthesis$ but $u \notin \llparenthesis b \rrparenthesis$, so $\llparenthesis a \rrparenthesis \neq \llparenthesis b \rrparenthesis$. In the other direction, if $\llparenthesis \text{-} \rrparenthesis$ is injective, and $a \neq \bot$, $\llparenthesis a \rrparenthesis \neq \llparenthesis \bot \rrparenthesis = \emptyset$, so there exists some $u \in \mathcal{U}^\sigma(A)$ such that $a \in u$. Therefore A is σ-spatial.

A map is a σ-Boolean isomorphism iff it is a bijective σ-Boolean homomorphism, so $\llparenthesis \text{-} \rrparenthesis$ is an isomorphism iff A is σ-spatial. □

Theorem 5. *(F, G, η, ϵ) is an adjunction making G a right adjoint to F. In fact, G maps into $\mathcal{PM}es$ and by restricting F and G to the categories where the unit and counit are isomorphisms, they define an adjoint equivalence $\sigma\text{-}\mathbf{BA}_{\mathrm{sp}}^{\mathrm{op}} \simeq \mathcal{PM}es$.*

Proof. By Lemma 4, $G(A) = (\mathcal{U}^\sigma(A), \mathcal{F}(A))$ is a measurable space. To show that $G(f)$ is well defined, first observe that the preimage of an ultrafilter is an ultrafilter.[4] To see that $f^{-1}(u)$ is a σ-filter if u is, suppose $(a_i)_{i \in \mathcal{N}}$ is a countable family of elements of A, such that $a_i \in f^{-1}(u)$. This implies

$$\begin{aligned} \forall i \in \mathcal{N}. f(a_i) \in u &\Rightarrow \bigwedge_{i=1}^{\infty} f(a_i) \in u \\ &\Rightarrow f\left(\bigwedge_{i=1}^{\infty} a_i\right) \in u \\ &\Rightarrow \bigwedge_{i=1}^{\infty} a_i \in f^{-1}(u). \end{aligned}$$

The next thing we do proves that $\llparenthesis \text{-} \rrparenthesis$ is a natural transformation and that $G(f)$ is measurable at the same time. The naturality diagram for $\llparenthesis \text{-} \rrparenthesis$, given $f : A \to B$ in $\sigma\text{-}\mathbf{BA}_{\mathrm{sp}}$ is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\llparenthesis \text{-} \rrparenthesis_A"'] & B \arrow[d, "\llparenthesis \text{-} \rrparenthesis_B"] \\ F(G(A)) \arrow[r, "F(G(f))"'] & F(G(B)). \end{tikzcd}$$

To show this diagram commutes, let $a \in A$:

$$\begin{aligned} F(G(f))(\llparenthesis a \rrparenthesis) &= G(f)^{-1}(\llparenthesis a \rrparenthesis) \\ &= \{u \in \mathcal{U}^\sigma(B) \mid G(f)(u) \in \llparenthesis a \rrparenthesis\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid f^{-1}(u) \in \llparenthesis a \rrparenthesis\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid a \in f^{-1}(u)\} \\ &= \{u \in \mathcal{U}^\sigma(B) \mid f(a) \in u\} \\ &= \llparenthesis f(a) \rrparenthesis. \end{aligned}$$

This also shows $G(f)$ is measurable, because every element of $\mathcal{F}(A)$ is of the form $\llparenthesis a \rrparenthesis$, and so its preimage is the measurable set $\llparenthesis f(a) \rrparenthesis \in \mathcal{F}(B)$. We also have that G is a contravariant functor by the usual rules for composition of inverse images, and $\llparenthesis \text{-} \rrparenthesis$ is a surjective natural transformation $\mathrm{Id} \Rightarrow FG$ by Lemma 4.

In the rest of the proof, we will use two identities twice, so we prove them now. The first is

$$\langle \text{-} \rangle^{-1} \left(\llparenthesis S \rrparenthesis_\Sigma \right) = S, \tag{6}$$

for all $S \in \Sigma$, where (X, Σ) is a measurable space, and $\llparenthesis \text{-} \rrparenthesis_\Sigma : \Sigma \to \mathcal{F}(\Sigma)$, treating Σ as a σ-Boolean algebra in its own right. We prove it as follows. Let $x \in X$ in the following

$$\begin{aligned} x \in \langle \text{-} \rangle^{-1} \left(\llparenthesis S \rrparenthesis_\Sigma \right) &\Leftrightarrow \langle x \rangle \in \llparenthesis S \rrparenthesis_\Sigma \\ &\Leftrightarrow S \in \langle x \rangle \\ &\Leftrightarrow x \in S, \end{aligned}$$

therefore $\langle \text{-} \rangle^{-1} \left(\llparenthesis S \rrparenthesis_\Sigma \right) = S$.

The second is

$$\llparenthesis \text{-} \rrparenthesis^{-1}(\langle u \rangle_{\mathcal{U}^\sigma(A)}) = u, \tag{7}$$

where u is a σ-ultrafilter on a σ-Boolean algebra A, and $\langle \text{-} \rangle_{\mathcal{U}^\sigma(A)}$ is the map taking an ultrafilter $u \in \mathcal{U}^\sigma(A)$ to the principal ultrafilter it defines in $\mathcal{U}^\sigma(\mathcal{F}(A))$. To prove it, let $a \in A$ in the following

$$\begin{aligned} a \in \llparenthesis \text{-} \rrparenthesis^{-1}(\langle u \rangle_{\mathcal{U}^\sigma(A)}) &\Leftrightarrow \llparenthesis a \rrparenthesis \in \langle u \rangle_{\mathcal{U}^\sigma(A)} \\ &\Leftrightarrow u \in \llparenthesis a \rrparenthesis \\ &\Leftrightarrow a \in u, \end{aligned}$$

and therefore $\llparenthesis \text{-} \rrparenthesis^{-1}(\langle u \rangle_{\mathcal{U}^\sigma(A)}) = u$.

We can now observe that $\langle \text{-} \rangle : (X, \Sigma) \to (\mathcal{U}^\sigma(\Sigma), \mathcal{F}(\Sigma))$ is measurable, because every element of $\mathcal{F}(\Sigma)$ is of the form $\llparenthesis S \rrparenthesis$ for some $S \in \Sigma$, and by (6) we have $\langle \text{-} \rangle^{-1} \left(\llparenthesis S \rrparenthesis \right) = S$. To show that $\langle \text{-} \rangle$ is a natural transformation, we only need to prove that

$$\begin{tikzcd} (X, \Sigma) \arrow[r, "f"] \arrow[d, "\langle \text{-} \rangle_X"'] & (Y, \Theta) \arrow[d, "\langle \text{-} \rangle_Y"] \\ G(F(X, \Sigma)) \arrow[r, "G(F(f))"'] & G(F(Y, \Theta)) \end{tikzcd}$$

commutes, if f is a measurable map.

Taking $x \in X$, we see

$$\begin{aligned} G(F(f))(\langle x \rangle) &= F(f)^{-1}(\langle x \rangle) \\ &= \{T \in \Theta \mid F(f)(T) \in \langle x \rangle\} \\ &= \{T \in \Theta \mid f^{-1}(T) \in \langle x \rangle\} \\ &= \{T \in \Theta \mid x \in f^{-1}(T)\} \\ &= \{T \in \Theta \mid f(x) \in T\} = \langle f(x) \rangle. \end{aligned}$$

We can now prove that F is a left adjoint to G, by proving that the following diagrams commute, for all $(X, \Sigma) \in \mathcal{M}es$ and $A \in \sigma\text{-}\mathbf{BA}$

$$\begin{tikzcd} F(X, \Sigma) & F(G(F(X, \Sigma))) \arrow[l, "F(\langle \text{-} \rangle)"'] \\ & F(X, \Sigma) \arrow[ul, "\mathrm{id}_\Sigma"] \arrow[u, "\llparenthesis \text{-} \rrparenthesis_\Sigma"'] \end{tikzcd} \qquad \begin{tikzcd} G(A) \arrow[r, "\langle \text{-} \rangle_{G(A)}"] \arrow[dr, "\mathrm{id}_{G(A)}"'] & G(F(G(A))) \arrow[d, "G(\llparenthesis \text{-} \rrparenthesis)"] \\ & G(A). \end{tikzcd}$$

[4] This fact is used in ordinary Stone duality, and is easier to prove using the isomorphism between ultrafilters and 2-valued measures, or the fact that every prime ideal is maximal in a Boolean ring.

By expanding the definitions of F and G, we see that commutativity of the left diagram is equivalent to (6), and commutativity of the right diagram is equivalent to (7). This finishes the proof that F is a left adjoint to G.

We already have that for all $(X,\Sigma) \in \mathcal{M}es$, $F(X,\Sigma)$ is σ-spatial, so we now prove that $G(A)$ is always σ-perfect, for any σ-Boolean algebra A. That is to say, we want to show that $\langle\text{-}\rangle_{G(A)} : \mathcal{U}^\sigma(A) \to \mathcal{U}^\sigma(\mathcal{F}(A))$ is a bijection. To show that it is injective, let $u, v \in \mathcal{U}^\sigma(A)$ and suppose that $\langle u\rangle = \langle v\rangle$. This means that for all $a \in A$, we have $(\!|a|\!) \in \langle u\rangle \Leftrightarrow (\!|a|\!) \in \langle v\rangle$. Now, $(\!|a|\!) \in \langle u\rangle \Leftrightarrow a \in u$, and $(\!|a|\!) \in \langle v\rangle \Leftrightarrow a \in v$, so we have shown that $u = v$.

To show that $\langle\text{-}\rangle_{G(A)}$ is surjective, let $u \in \mathcal{U}^\sigma(\mathcal{F}(A)) = G(F(G(A)))$. We can produce an element of $\mathcal{U}^\sigma(A)$ by taking $G((\!|\text{-}|\!))(u)$, so we will be able to show surjectivity if we show that $\langle G((\!|\text{-}|\!))(u)\rangle_{G(A)} = u$. Every element of $\mathcal{F}(A)$ is of the form $(\!|a|\!)$ for some $a \in A$, so let $(\!|a|\!) \in \mathcal{F}(A)$ in the following

$$\begin{aligned}(\!|a|\!) \in \langle G((\!|\text{-}|\!))(u)\rangle_{G(A)} &\Leftrightarrow G((\!|\text{-}|\!))(u) \in (\!|a|\!)\\ &\Leftrightarrow a \in G((\!|\text{-}|\!))(u)\\ &\Leftrightarrow a \in (\!|\text{-}|\!)^{-1}(u)\\ &\Leftrightarrow (\!|a|\!) \in u.\end{aligned}$$

We have therefore proven that $\langle G((\!|\text{-}|\!)(u))\rangle_{G(A)} = u$ and so $G(A)$ is σ-perfect.

We only required $\langle\text{-}\rangle$ to be a bijection in the definition of a σ-perfect space. We now show that if it is a bijection, it is an isomorphism of measurable spaces. To show this, we only need to show that the image of a measurable set is measurable. If we apply $\langle\text{-}\rangle$ (taking the image rather than applying to an element) to each side of (6) we get that for all $S \in \Sigma$, $(\!|S|\!) = \langle\text{-}\rangle\left(\langle\text{-}\rangle^{-1}\left((\!|S|\!)\right)\right) = \langle\text{-}\rangle\,(S)$. As $(\!|S|\!) \in \mathcal{F}(\Sigma)$, we have shown that the image of a measurable set is measurable.

Therefore we can characterize $\mathcal{PM}es$ as the full subcategory of $\mathcal{M}es$ on which $\langle\text{-}\rangle$ is an isomorphism, and σ-$\mathbf{BA}_{\mathrm{sp}}$ as the full subcategory of σ-$\mathbf{BA}$ such that $(\!|\text{-}|\!)$ is an isomorphism. Therefore the restriction of F and G to these full subcategories define an adjoint equivalence $\mathcal{PM}es \simeq \sigma\text{-}\mathbf{BA}_{\mathrm{sp}}^{\mathrm{op}}$, *i.e.* this duality arises from "unity of opposites" [32, Part 0, Proposition 4.2]. □

We provide a proof of the following fact, well known, and used implicitly by Halmos in [38].

Lemma 17. *In a Stone space, zero-sets are the same as countable unions of clopen sets (called σ-closed sets). Therefore the Baire σ-field can also be defined as the σ-field generated by the clopens.*

Proof. Let X be a Stone space, and $C = \bigcap_{i=1}^\infty G_i$ a σ-closed set. As each G_i is clopen, χ_{G_i} is a continuous function, as is $\chi_{X\setminus G_i}$. We can then define

$$f = \sum_{i=1}^{\infty} 2^{-i}\chi_{X\setminus G_i}.$$

This sum is absolutely convergent in norm, and therefore f is a continuous function. We also have that $f(x) = 0$ iff $x \in \bigcap_{i=1}^\infty G_i = C$, so C is a zero-set.

For the reverse implication, let Z be a zero-set in X, defined by $f : X \to \mathbb{R}$. We know that $\{0\} = \bigcap_{i=1}^\infty (-2^{-i}, 2^{-i})$ in $\mathbb{R}$, so $Z = f^{-1}(0) = \bigcap_{i=1}^\infty f^{-1}((-2^{-i}, 2^{-i}))$, and so we can define $U_i = f^{-1}((-2^{-i}, 2^{-i}))$. We will define a family of clopens $(G_i)_{i\in\mathcal{N}}$ such that $Z \subseteq G_i \subseteq U_i$. If we succeed, then we have $Z \subseteq \bigcap_{i=1}^\infty G_i \subseteq \bigcap_{i=1}^\infty U_i = Z$ and will have proven that Z is σ-closed. We first observe that for each i, and each $x \in Z$, we have by the continuity of f at x and the fact that clopens are a base that there exists a clopen $G_{i,x}$ such that $x \in G_{i,x} \subseteq f^{-1}((-2^{-i}, 2^{-i})) = U_i$. Fixing i, we see that $G_{i,x}$ is a clopen cover of Z, and since Z is closed, hence compact, there exists a finite subcover. If we take the union of this subcover, as the finite union of clopen sets, it is clopen, so we have a G_i such that $Z \subseteq G_i$ and $G_i \subseteq U_i$. This was all we needed to prove. □

Proposition 7 (Halmos)**.** *The functor U has a left adjoint H, given on objects by*

$$H(X) = \mathcal{B}a(2^X),$$

where 2^X is given the product topology. This is also a left adjoint to the restriction of U to σ-$\mathbf{BA}_{\mathrm{sp}}$.

Proof. In [38, §23, Theorem 14] Halmos defines a map η_X (as $h^* \circ h$ in his notation) from $X \to U(\mathcal{B}a(2^X))$, explicitly

$$\eta_X(x) = \{f \in 2^X \mid f(x) = 1\}.$$

Halmos then defines for each map $f : X \to U(A)$, A being a σ-Boolean algebra, a σ-Boolean homomorphism $\tilde{f} : \mathcal{B}a(2^X) \to A$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U\mathcal{B}a(2^X) \arrow[d, "U\tilde{f}"] \\ & UA \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}a(2^X) \arrow[d, dotted, "\tilde{f}"] \\ A \end{tikzcd}$$

If $g : \mathcal{B}a(2^X) \to A$ is a σ-homomorphism such that the above diagram commutes with g in place of $\tilde{f}$, we know that $\tilde{f}$ and g agree on the image of η_X. As the elements $\{\eta_X(x)\}_{x\in X}$ generate the clopen sets of 2^X, we have that $\tilde{f}$ and g agree on the clopens, because they are Boolean homomorphisms. As 2^X is a Stone space, the clopens generate the Baire σ-field (Lemma 17), so $\tilde{f}$ and g agree on $\mathcal{B}a(2^X)$ because they are σ-homomorphisms. This proves the uniqueness of $\tilde{f}$, as required for the universal property. Therefore, by [33, IV.1 Theorem 2 (ii)] there exists a functor H such that $H(X) = \mathcal{B}a(2^X)$ and η_X is a natural transformation making H a left adjoint to U.

By Proposition 6, as 2^X is a compact Hausdorff space $\mathcal{B}a(2^X)$ is a σ-perfect Boolean algebra, so H, as defined above, is also a left adjoint to the restriction of U to σ-$\mathbf{BA}_{\mathrm{sp}}$. □

Lemma 8. *A set $K \subseteq A$ generates A iff the universal map $\tilde{i} : H(K) \to A$:*

$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[dr, "i"'] & U(H(K)) \arrow[d, "U\tilde{i}"] & H(K) \arrow[d, dotted, "\tilde{i}"] \\ & U(A) & A \end{tikzcd}$$

is surjective, where i is the inclusion morphism.

Proof. If K generates A, then for each $a \in A$ we have an expression in the language of σ-Boolean algebras expressing a in terms of elements of K. This can be reinterpreted as an element $b \in H(K)$, with the σ-Boolean operations being those in $H(K)$ instead. Then as $\tilde{i}$ preserves the σ-Boolean operations, $\tilde{i}(b) = a$, so $\tilde{i}$ is surjective.

If, on the other hand, $\tilde{i}$ is assumed to be surjective, then given $a \in A$, we have some element $b \in H(K)$ such that $\tilde{i}(b) = a$. As $H(K)$ is generated by K (K generates the clopens of $H(K)$, and the clopens of a Stone space generate the Baire σ-field by Lemma 17), there is a σ-Boolean expression for b in terms of elements of K. Since $\tilde{i}$ preserves the σ-Boolean operations, this shows that a can be expressed in terms of elements of K, and therefore K generates A. $\square$

Proposition 9.

(i) Let A and B be σ-Boolean algebras, (K, R) a presentation of A, and $f : K \to B$ a function. There exists a σ-homomorphism $g : A \to B$ such that $g|_K = f$ iff $\tilde{f}(r) = \bot$ for all $r \in R$.

(ii) Let (X, Σ) and (Y, Θ) be measurable spaces, where (X, Σ) is σ-perfect, and let (K, R) be a presentation of (X, Σ). Let $f : K \to \Theta$ be a function. There exists a measurable map $g : (Y, \Theta) \to (X, \Sigma)$ such that $g^{-1}|_K = f$ iff $\tilde{f}(r) = \emptyset$ for all $r \in R$.

Proof. Part (ii) is a direct consequence of part (i) and Theorem 5, so we only give the proof of part (i).

- Only if: If there exists a σ-homomorphism $g : A \to B$ extending f, then the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[dr, "i"] \arrow[ddr, "f"'] & H(K) \arrow[d, "\tilde{i}"] \\ & A \arrow[d, "g"] \\ & B, \end{tikzcd}$$

  and so by the uniqueness part of the adjunction, $g \circ \tilde{i} = \tilde{f}$. By the definition of a presentation, $\tilde{i}(r) = \bot$ for all $r \in R$, so $\tilde{f}(r) = \bot$ holds also.
- If: Suppose $\tilde{f}(r) = \bot$ for all $r \in R$. Given $a \in A$, as $\tilde{i}$ is surjective (Lemma 8), there exists $b \in H(K)$ such that $\tilde{i}(b) = a$. Define $g(a) = \tilde{f}(b)$. We prove this is well-defined as follows. If $\tilde{i}(b') = a$ as well, then $\tilde{i}(b \triangle b') = \bot$, so there exists a countable set of $(r_i)_{i \in I}$, such that each $r_i \in R$, such that $b \triangle b' \subseteq \bigwedge_{i \in I} r_i$ (by the definition of a presentation). We then have that $\tilde{f}(\bigwedge_{i \in I} r_i) = \bot$ by the assumption that $\tilde{f}$ maps elements of R to $\bot$, so $\tilde{f}(b) = \tilde{f}(b')$, proving well-definedness of g. It is easy to show that g preserves negation and countable joins using the corresponding properties of $\tilde{f}$ and $\tilde{i}$. We also have that for a generator $a \in K$, $h(a) = \tilde{f}(\eta_K(a)) = f(a)$, as required.

$\square$

Lemma 10. *Let (X, Σ) be a σ-perfect measurable space, $K \subseteq \Sigma$ a set of generators with inclusion morphism $i : K \to \Sigma$, and $(r_j)_{j \in J}$ a countable set of relations,* i.e. *elements of $H(K)$ such that $\tilde{i}(r_j) = \emptyset$. The following condition implies that $(K, (r_j)_{j \in J})$ is a presentation of (X, Σ): For all $u \in \mathcal{U}^\sigma(H(K))$ such that $\forall j \in J. r_j \notin u$, we have that there exists $x \in X$ such that $\tilde{i}^{-1}(\langle x \rangle) = u$.*

Proof. Let $(K, (r_j)_{j \in J})$ satsify the hypotheses of the lemma. In order to show that $(K, (r_j)_{j \in J})$ is a presentation, we only need to show that $(r_j)_{j \in J}$ generates the σ-ideal $\tilde{i}^{-1}(\emptyset)$. Defining $r = \bigvee_{j \in J} r_j$, this is equivalent to showing that if $a \in H(K)$ is such that $\tilde{i}(a) = \emptyset$, then $a \leq r$. So suppose that $\tilde{i}(a) = \emptyset$, and suppose for a contradiction that $a \not\leq r$. As $H(K)$ is σ-spatial (Proposition 7), there is a σ-ultrafilter u containing $a \setminus r$, and therefore such that $a \in u$ and $r \notin u$. Therefore there exists an $x \in X$ such that $\tilde{i}^{-1}(\langle x \rangle) = u$, and so $a \in \tilde{i}^{-1}(\langle x \rangle)$. This is equivalent to $x \in \tilde{i}(a)$, which contradicts $\tilde{i}(a)$ being empty. As we have reached a contradiction, it must be the case that $a \leq r$ after all. $\square$

In the following, we use a well-known lemma that we did not state in the main text. It is a basic fact about symmetric differences and ultrafilters.

Lemma 18. *Let A be a Boolean algebra, and u an ultrafilter on it. If $a, b \in A$, $a \in u$ and $a \triangle b \notin u$, we have $b \in u$.*

Proof. Let u, a, b be as described above. As u is an ultrafilter, $\neg(a \triangle b) \in u$. We have

$$\neg(a \triangle b) = \neg((a \wedge \neg b) \vee (b \wedge \neg a)) = (\neg a \vee b) \wedge (\neg b \vee a)$$

As u is up-closed, we have $\neg a \vee b \in u$, and as it is closed under finite meets this implies that $(\neg a \vee b) \wedge a \in u$. Now, $(\neg a \vee b) \wedge a = b \wedge a$, so up-closedness of u gives us $b \in u$. $\square$

Lemma 11. *The above (K, R), as in* (1) *and* (2)*, define a presentation of* $([0,1], \mathcal{B}o([0,1]))$.

Proof. The proof has three steps, as we use Lemma 10.

- K generates $\mathcal{B}o([0,1])$:
  We have $K \subseteq \mathcal{B}o([0,1])$ because each $[r,1] \in K$ is a closed, and therefore Borel, set. We can define $[r,1]$ for any $r \in [0,1]$ as $\bigcap_{s \in [0,r) \cap \mathbb{Q}} [s,1]$, so it is in the σ-field generated by K. We can then define $[0,r)$ for $r \in [0,1]$ as $\neg[r,1]$, and we can define $(r,1]$ for $r \in [0,1]$ as $\bigcup_{s \in (r,1] \cap \mathbb{Q}} [r,1]$. Then every open interval (r,s) for $r, s \in [0,1]$ as $[0,s) \cap (r,1]$. Since every open set is a

countable union of open intervals (including $[0,r)$ and $(r,1]$ as open intervals), we have that every open set is in the σ-field generated by K, and therefore all the Borel sets are in the σ-field generated by K. Therefore K generates $\mathcal{B}o([0,1])$.

- For all $r \in R$, $\tilde{i}(r) = \emptyset$:
For the first kind of relation, let $r < s \in \mathbb{Q}_0$. We want to show that $\tilde{i}((B_r \cap B_s)\triangle B_s) = 0$. We have
$$\tilde{i}((B_r \cap B_s)\triangle B_s) = ([r,1] \cap [s,1])\triangle[s,1] = \emptyset$$
because $[r,1] \cap [s,1] = [s,1]$.
For the second kind of relation, let $s \in \mathbb{Q}_0$, and observe
$$\tilde{i}\left(\left(\bigwedge_{r<s} B_r\right)\triangle B_s\right) = \left(\bigcap_{r<s}[r,1]\right)\triangle[s,1] = \emptyset.$$
Therefore all relations hold.
- For each σ-ultrafilter $u \in \mathcal{U}^\sigma(H(K))$ such that $\forall r \in R. r \notin u$, there exists $x \in [0,1]$ such that $\tilde{i}^{-1}(\langle x\rangle) = u$:
Let $u \in \mathcal{U}^\sigma(H(K))$ be a σ-ultrafilter respecting the relations in R, *i.e.* such that for all $r \in R$, $r \notin u$. Define
$$x = \sup\{r \in \mathbb{Q}_0 \mid B_r \in u\},$$
and we see that $x \in [0,1]$. We want to show that $\tilde{i}^{-1}(\langle x\rangle) = u$, *i.e.* for all $T \in H(K)$, $x \in \tilde{i}(T)$ iff $T \in u$. To do this, we use "structural induction" on T, as an element of $H(K)$.
  - T is a generator B_r of $H(K)$:
  We want to show $B_r \in u \Leftrightarrow x \in \tilde{i}(B_r)$. We can start by observing that $\tilde{i}(B_r) = [r,1]$ and so
  $$\begin{aligned} x \in \tilde{i}(B_r) &\Leftrightarrow x \in [r,1] \\ &\Leftrightarrow x \geq r \\ &\Leftrightarrow \sup\{s \in \mathbb{Q}_0 \mid B_s \in u\} \geq r. \end{aligned}$$
  If $B_r \in u$, we have that $r \in \{s \in \mathbb{Q}_0 \mid B_s \in u\}$, so
  $$x = \sup\{s \in \mathbb{Q}_0 \mid B_s \in u\} \geq r,$$
  so we have shown one direction, that $B_r \in u \Rightarrow x \in \tilde{i}(B_r)$. For the other direction, we distinguish two cases:
    * For all $r \in \mathbb{Q}_0$, $x = \sup\{s \in \mathbb{Q}_0 \mid B_s \in u\} > r$ implies $B_r \in u$:
    We know that if the supremum of some set strictly exceeds a number, then there must be an element of that set strictly exceeding that number. So there exists $s \in \mathbb{Q}_0$ such that $B_s \in u$ and $s > r$. Therefore $s \geq r$ and so the relation $(B_s \wedge B_r)\triangle B_s \notin u$. Taken together, $B_s \in u$ and $(B_s \wedge B_r)\triangle B_s \notin u$ imply $B_r \in u$ by Lemma 18.
    * For all $r \in \mathbb{Q}_0$, $x = \sup\{s \in \mathbb{Q}_0 \mid B_s \in u\} = r$ implies $B_r \in u$:
    By the previous item, we have that for all $s \in \mathbb{Q}_0$ such that $s < r$, $B_s \in u$, and therefore $\bigwedge_{s<r} B_s \in u$ as u is a σ-filter. As $\left(\bigwedge_{s<r} B_s\right)\triangle B_r \notin u$, we can apply Lemma 18 to deduce $B_r \in u$.
  - Inductive step for $\neg$:
  Suppose we have $T \in H(K)$ such that $x \in \tilde{i}(T) \Leftrightarrow T \in u$. Then
  $$\begin{aligned} \neg T \in u \Leftrightarrow T \notin u \Leftrightarrow x \notin \tilde{i}(T) \Leftrightarrow x \in \neg\tilde{i}(T) \\ \Leftrightarrow x \in \tilde{i}(\neg T). \end{aligned}$$
  - Inductive step for $\bigwedge$:
  Suppose $(T_i)_{i\in\mathcal{N}}$ such that $T_i \in H(K)$ and $x \in \tilde{i}(T_i) \Leftrightarrow T_i \in u$ for all $i \in \mathcal{N}$. Then
  $$\begin{aligned} x \in \tilde{i}\left(\bigwedge_{i=1}^{\infty} T_i\right) &\Leftrightarrow x \in \bigwedge_{i=1}^{\infty} \tilde{i}(T_i) \\ &\Leftrightarrow \forall i \in \mathcal{N}. x \in \tilde{i}(T_i) \\ &\Leftrightarrow \forall i \in \mathcal{N}. T_i \in u \Leftrightarrow \bigwedge_{i=1}^{\infty} T_i \in u, \end{aligned}$$
  because u is a σ-ultrafilter. ☐

Proposition 12. *F defines a a functor* **Markov** $\to$ **AumannSp**$^{\mathrm{op}}$.

Proof. We first show that each L_r defines a map $\Sigma \to \Sigma$. For any $S \in \Sigma$, we have
$$\begin{aligned} L_r(S) &= \{x \in X \mid \theta(x)(S) \geq r\} \\ &= \{x \in X \mid (p_S \circ \theta)(x) \geq r\} \\ &= \{x \in X \mid (p_S \circ \theta)(x) \in [r,1]\} \\ &= (p_S \circ \theta)^{-1}([r,1]) \end{aligned}$$
As this is the preimage of a measurable set under a measurable map, it gives an element of Σ. We now verify that $(\Sigma, (L_r)_{r\in\mathbb{Q}_0})$ satisfies all eight σ-Aumann algebra axioms.

- AA1 – $X \subseteq L_0(S)$ for all $S \in \Sigma$:
We have $L_0(S) = (p_S \circ \theta)^{-1}([0,1]) = \theta^{-1}(p_S^{-1}([0,1]))$. Now $p_S^{-1}([0,1]) = \mathcal{G}(X)$, so $L_0(S) = \theta^{-1}(\mathcal{G}(X)) = X$.
- AA2 – $L_r(\emptyset) \subseteq \emptyset$ for $r > 0$:
We have $L_r(\emptyset) = \theta^{-1}(p_\emptyset^{-1}([r,1]))$. We also have $\nu(\emptyset) = 0$ for all $\nu \in \mathcal{G}(X)$, so $p_\emptyset^{-1}([r,1]) = \emptyset$. Therefore $L_r(\emptyset) = \theta^{-1}(\emptyset) = \emptyset$.
- AA3 – $L_r(S) \subseteq \neg L_s(\neg S)$ for $S \in \Sigma$, $r, s \in \mathbb{Q}_0$ such that $r + s > 1$:
We have that $L_r(S) \subseteq \neg L_s(\neg S)$ iff $L_r(S) \cap L_s(\neg S) = \emptyset$. Suppose, for a contradiction, that $x \in L_r(S) \cap L_s(\neg S)$. Then $\theta(x)(S) \geq r$ and $\theta(x)(\neg S) \geq s$, so by the additivity of measures
$$\theta(x)(X) \geq r + s > 1,$$
a contradiction.
- AA4 – $L_r(S \cap T) \cap L_s(S \setminus T) \subseteq L_{r+s}(S)$, where $S \in \Sigma$, $r, s \in \mathbb{Q}_0$ and $r + s \leq 1$:
Consider
$$\begin{aligned} p_{S\cap T}^{-1}([r,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(S \cap T) \geq r\} \\ p_{S\setminus T}^{-1}([s,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(S \setminus T) \geq s\} \\ p_S^{-1}([r+s,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(S) \geq r + s\} \end{aligned}$$

The additivity of measures implies that if $\nu(S \cap T) \geq r$ and $\nu(S \setminus T) \geq s$, then

$$\nu(S) = \nu(S \cap T) + \nu(S \setminus T) \geq r + s,$$

so we have shown $p_{S\cap T}^{-1}([r,1]) \cap p_{S\setminus T}^{-1}([s,1]) \subseteq p_S^{-1}([r+s,1])$, which implies AA4.

- AA5 – $\neg L_r(S \cap T) \cap \neg L_s(S \setminus T) \subseteq \neg L_{r+s}(S)$, where $S \in \Sigma$, $r, s \in \mathbb{Q}_0$ and $r + s \leq 1$:
We see

$$\begin{aligned}\neg p_{S\cap T}^{-1}([r,1]) &= \neg\{\nu \in \mathcal{G}(X) \mid \nu(S \cap T) \geq r\} \\ &= \{\nu \in \mathcal{G}(X) \mid \nu(S \cap T) < r\} \\ \neg p_{S\setminus T}^{-1}([s,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(S \setminus T) < s\} \\ \neg p_S^{-1}([r+s,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(S) < r + s\}.\end{aligned}$$

Using the additivity of the measures, $\nu(S \cap T) < r$ and $\nu(S \setminus T) < s$ imply $\nu(S) = \nu(S \cap T) + \nu(S \setminus T) < r + s$, so AA5 follows in a similar manner to the end of AA4.

- AA6 – $S \subseteq T \Rightarrow L_r(S) \subseteq L_r(T)$ where $S, T \in \Sigma$ and $r \in \mathbb{Q}_0$:
Let $S, T \in \Sigma$ with $S \subseteq T$. We see that

$$\begin{aligned}p_S^{-1}([r,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(S) \geq r\} \\ p_T^{-1}([r,1]) &= \{\nu \in \mathcal{G}(X) \mid \nu(T) \geq r\}.\end{aligned}$$

If $\nu(S) \geq r$, we have $\nu(T) \geq \nu(S) \geq r$, so $p_S^{-1}([r,1]) \subseteq p_T^{-1}([r,1])$, and therefore $L_r(S) \subseteq L_r(T)$.

- AA7 – $\bigcap\limits_{r<s} L_r(S) = L_s$, where $S \in \Sigma$ and $s \in \mathbb{Q}_0$:
We reason as follows:

$$\begin{aligned}\bigcap_{r<s} L_r(S) &= \bigcap_{r<s} (p_S \circ \theta)^{-1}([r,1]) \\ &= (p_S \circ \theta)^{-1}\left(\bigcap_{r<s}[r,1]\right) \\ &= (p_S \circ \theta)^{-1}([s,1]) \\ &= L_s(S).\end{aligned}$$

- AA8 – $\bigcap\limits_{i=1}^{\infty} L_r(S_i) = \emptyset$ for $S_i \in \Sigma$ where $S_1 \geq S_2 \geq \cdots$ and $\bigcap\limits_{i=1}^{\infty} S_i = \emptyset$ and $r \in \mathbb{Q}_0$, $r \neq 0$:
We have

$$\begin{aligned}\bigcap_{i=1}^{\infty} L_r(S_i) &= \bigcap_{i=1}^{\infty} \theta^{-1}(p_{S_i}^{-1}([r,1])) \\ &= \theta^{-1}\left(\bigcap_{i=1}^{\infty} p_{S_i}^{-1}([r,1])\right)\end{aligned}$$

For each $\nu \in \mathcal{G}(X)$, by [42, §9 Theorem E] countable additivity implies that $\lim_{i\to\infty} \nu(S_i) = 0$. As $r > 0$, there exists a $j \in \mathcal{N}$ such that for all $i \geq j$ $\nu(S_i) < r$, and so

$$\nu \notin \bigcap_{i=1}^{\infty} p_{S_i}^{-1}([r,1]).$$

As this applies for an arbitrary element of $\mathcal{G}(X)$, we have shown that

$$\bigcap_{i=1}^{\infty} p_{S_i}^{-1}([r,1]) = \emptyset.$$

Therefore, as $\theta^{-1}(\emptyset) = \emptyset$ we have shown AA8.

As $F(X, \Sigma)$ is always a σ-spatial Boolean algebra, we have that $F(X, \Sigma, \theta)$ is a σ-spatial Aumann algebra.

Now that we have shown that this works on objects, we turn our attention to maps. Let $f : (X, \Sigma, \theta) \to (Y, \Theta, \lambda)$ be a morphism of Markov processes. We want to show $F(f)$ is a morphism of σ-Aumann algebras. We have already that $F(f)$ is a σ-Boolean homomorphism, so we only need to show that $F(f)(M_r(T)) = L_r(F(f)(T))$ for each $T \in \Theta$, where $F(X, \Sigma, \theta) = (\Sigma, L_r)$ and $F(Y, \Theta, \lambda) = (\Theta, M_r)$. That is to say, we want to show

$$f^{-1}(\lambda^{-1}(p_T^{-1}([r,1]))) = \theta^{-1}(p_{f^{-1}(T)}([r,1])). \tag{8}$$

We prove it as follows:

$$\begin{aligned}x \in f^{-1}(\lambda^{-1}(p_T^{-1}([r,1]))) &\Leftrightarrow p_T(\lambda(f(x))) \in [r,1] \\ &\Leftrightarrow \lambda(f(x))(T) \geq r \\ &\Leftrightarrow \theta(x)(f^{-1}(T)) \geq r \\ &\Leftrightarrow p_{f^{-1}(T)}(\theta(x)) \in [r,1] \\ &\Leftrightarrow x \in \theta^{-1}(p_{f^{-1}(T)}^{-1}([r,1])).\end{aligned}$$

The fact that F preserves identity morphisms and composition follows from the fact that F does so as a functor from $\mathcal{M}es \to \sigma\text{-}\mathbf{BA}_{\text{sp}}^{\text{op}}$. □

To prove that each σ-Aumann algebra gives rise to a Markov process, we will use some lemmas not appearing in the main article.

Lemma 19. *Let $(A, (L_r)_{r\in\mathbb{Q}_0})$ be a σ-Aumann algebra. If $a \in A$, $r, s \in \mathbb{Q}_0$ with $r < s$, then $L_s(a) \leq L_r(a)$.*

Proof. Let $a \in A$, $r, s \in \mathbb{Q}_0$ with $r < s$. Then $s - r \in \mathbb{Q}_0$ and $s - r > 0$, and $r + (s - r) = s \leq 1$. So

$$\begin{aligned}\neg L_r(a) &= \neg L_r(a) \wedge \neg\bot \\ &= \neg L_r(a) \wedge \neg L_{s-r}(\bot) && \text{AA2} \\ &= \neg L_r(a \wedge \top) \wedge \neg L_{s-r}(a \wedge \neg\top) \\ &\leq \neg L_{r+(s-r)}(a) && \text{AA5, } b = \top \\ &= \neg L_s(a).\end{aligned}$$

Therefore $L_s(a) \leq L_r(a)$. □

Lemma 20. *Let $a, b, c \in [0,1]$.*

(i) Suppose that $\forall r, s \in \mathbb{Q}_0$, $a < r$ and $b < s$ implies $c < r + s$. Then $c \leq a + b$.

(ii) Suppose that $\forall r, s \in \mathbb{Q}_0$, $a \geq r$ and $b \geq s$ implies $c \geq r + s$. Then $c \geq a + b$.

Proof.

(i) If $a = 1$ or $b = 1$, then $c \leq 1 \leq a + b$ so we reduce to the case that $a < 1$ and $b < 1$. Assume for a contradiction that $c > a + b$, and let $\epsilon = c - (a + b) > 0$. There exist rationals $r, s \in \mathbb{Q}_0$ such that $r > a$ and $r - a < \frac{\epsilon}{2}$, and $s > b$ and $s - b < \frac{\epsilon}{2}$. By the hypothesis of the lemma, $c < r + s$. However,

$$r + s - (a + b) < \epsilon = c - (a + b),$$

so $r + s < c$, a contradiction.

(ii) Suppose for a contradiction that $c < a + b$, and let $\epsilon = a + b - c > 0$. Then there exist $r, s \in \mathbb{Q}_0$ such that $a \geq r$ and $a - r < \frac{\epsilon}{2}$, and $b \geq s$ and $b - s < \frac{\epsilon}{2}$. By the hypothesis of the lemma, $c \geq r + s$, but

$$a + b - (r + s) < \epsilon = a + b - c,$$

so $c < r + s$, a contradiction. □

Proposition 13. *If $(A, (L_r)_{r \in \mathbb{Q}_0})$ is a σ-Aumann algebra, $(\mathcal{U}^\sigma(A), \mathcal{F}(A), \theta)$ is a σ-perfect Markov process.*

Proof. There are three steps – showing that θ_a is defined correctly, that $\theta(u)$ is a σ-additive measure for all $u \in \mathcal{U}^\sigma(A)$, and that $\theta : \mathcal{U}^\sigma(A) \to \mathcal{G}(\mathcal{U}^\sigma(A))$ is measurable.

- We use Proposition 9, taking $(X, \Sigma) = ([0, 1], \mathcal{B}o([0, 1])$, $(Y, \Theta) = (\mathcal{U}^\sigma(A), \mathcal{F}(A))$ and (K, R) is the presentation of $[0, 1]$ from Lemma 11. We define $f_a : K \to \mathcal{F}(A)$ as

$$f_a([r, 1]) = (\!|L_r(a)|\!).$$

  To use Proposition 9, we need to show that $\tilde{f}(r) = 0$ for all $r \in R$ (see (2)).

  - $\tilde{f}((B_r \wedge B_s) \triangle B_s) = \emptyset$ for $r < s$ in $\mathbb{Q}_0$:

$$\begin{aligned}\tilde{f}((B_r \wedge B_s) \triangle B_s) &= (f([r, 1]) \wedge f([s, 1])) \triangle f([s, 1]) \\ &= ((\!|L_r(a)|\!) \wedge (\!|L_s(a)|\!)) \triangle (\!|L_s(a)|\!) \\ &= (\!|(L_r(a) \wedge L_s(a)) \triangle L_s(a)|\!),\end{aligned}$$

    which is equal to $\emptyset$ because $L_r(a) \wedge L_s(a) = L_s(a)$ (Lemma 19).

  - $\tilde{f}\left(\left(\bigwedge_{r<s} B_r\right) \triangle B_s\right) = \emptyset$ for all $s \in \mathbb{Q}_0$:

$$\begin{aligned}\tilde{f}\left(\left(\bigwedge_{r<s} B_r\right) \triangle B_s\right) &= \left(\bigwedge_{r<s} f_a([r, 1])\right) \triangle f_a([s, 1]) \\ &= \left(\bigwedge_{r<s} (\!|L_r(a)|\!)\right) \triangle (\!|L_s(a)|\!) \\ &= (\!|\left(\bigwedge_{r<s} L_r(a)\right) \triangle L_s(a)|\!)\end{aligned}$$

    and by AA7 this is $(\!|\bot|\!) = \emptyset$.

  Therefore we have shown that for each $a \in A$ there exists a measurable map $\theta_a : \mathcal{U}^\sigma(A) \to [0, 1]$ such that $\theta_a^{-1}([r, 1]) = (\!|L_r(a)|\!)$, by applying Proposition 9 to f_a.

- $\theta(u)$ is σ-additive for all $u \in \mathcal{U}^\sigma(A)$:
  Recall that $\theta(u)$ is defined as $\theta(u)((\!|a|\!)) = \theta_a(u)$. We do the proof in two steps — We first show that $\theta(u)$ is finitely additive, prove countable additivity as a separate step.

  - $\theta(u)$ finitely additive:

    Let $a, b \in A$ such that $a \wedge b = \bot$. We want to show that $\theta(u)(a \vee b) = \theta(u)(a) + \theta(u)(b)$. We do this in two steps.

    * $\theta(u)(a \vee b) \leq \theta(u)(a) + \theta(u)(b)$:
      First, observe that, for $r \in \mathbb{Q}_0$

$$\begin{aligned}\theta(u)(a) < r &\Leftrightarrow \theta_a(u) < r \\ &\Leftrightarrow \theta_a(u) \in [0, r) \\ &\Leftrightarrow \theta_a(u) \notin [r, 1] \\ &\Leftrightarrow u \notin \theta_a^{-1}([r, 1]) \\ &\Leftrightarrow u \notin (\!|L_r(a)|\!) \\ &\Leftrightarrow L_r(a) \notin u \\ &\Leftrightarrow \neg L_r(a) \in u.\end{aligned}$$

      Similarly $\theta(u)(b) < s \Leftrightarrow \neg L_s(b) \in u$ and $\theta(u)(a \vee b) < t \Leftrightarrow \neg L_t(a \vee b) \in u$.
      We now show that if $r, s \in \mathbb{Q}_0$ and $\theta(u)(a) < r$ and $\theta(u)(b) < s$, then $\theta(u)(a \vee b) < r + s$.
      If $r + s > 1$, then $\theta(u)(a \vee b) \leq 1 < r + s$, so we can reduce to the case that $r + s \leq 1$. By the previous discussion, we have $\neg L_r(a) \in u$ and $\neg L_s(b) \in u$, so as u is an ultrafilter, $\neg L_r(a) \wedge \neg L_s(b) \in u$. We can apply AA5 with $a = a \vee b$ and $b = a$ to deduce that

$$\neg L_r((a \vee b) \wedge a) \wedge L_r((a \vee b) \wedge \neg a) \leq L_{r+s}(a \vee b)$$

      As $a \wedge b = \bot$, the previous statement is equivalent to

$$\neg L_r(a) \wedge \neg L_s(b) \leq \neg L_{r+s}(a \vee b),$$

      so by the up-closedness of ultrafilters, $\neg L_{r+s}(a \vee b) \in u$, and so $\theta(u)(a \vee b) > r + s$.
      We can therefore apply Lemma 20 (i) to conclude that $\theta(u)(a \vee b) \leq \theta(u)(a) + \theta(u)(b)$.

    * $\theta(u)(a \vee b) \geq \theta(u)(a) + \theta(u)(b)$:
      Observe that this time

$$\theta(u)(a) \geq r \Leftrightarrow \theta_a(u) \geq r \Leftrightarrow L_r(a) \in u$$

      and likewise for b and $a \vee b$.
      We will show that if $r, s \in \mathbb{Q}_0$ such that $\theta(u)(a) \geq r$ and $\theta(u)(b) \geq s$ then $\theta(u)(a \vee b) \geq r + s$.
      If $\theta(u)(a) \geq r$ and $\theta(u)(b) \geq s$, then $L_r(a) \in u$ and $L_s(b) \in u$. So $L_r(a) \wedge L_s(b) \in u$. We first show that $r + s \leq 1$. Suppose for a contradiction that $r + s > 1$. Then AA3 shows that $L_r(a) \leq \neg L_s(\neg a)$, and as $b \leq \neg a$ (disjointness), AA6 shows that $L_s(b) \leq L_s(\neg a) \in u$. So $L_r(a) \leq \neg L_s(\neg a) \leq \neg L_s(b)$, and therefore $L_r(a) \wedge L_s(b) = \bot$, contradicting $L_r(a) \wedge L_s(b) \in u$. So we must have $r + s \leq 1$.
      We can therefore apply AA4 with $a = a \vee b$ and $b = a$ to get

$$L_r((a \vee b) \wedge a) \wedge L_s((a \vee b) \wedge \neg a) \leq L_{r+s}(a \vee b)$$

By disjointness of a and b, this is equivalent to

$$L_r(a) \wedge L_s(b) \leq L_{r+s}(a \vee b)$$

so by up-closedness of ultrafilters, we have $L_{r+s}(a \vee b) \in u$, and therefore $\theta(u)(a \vee b) \geq r + s$. We then apply Lemma 20 (ii) to conclude $\theta(u)(a \vee b) \geq \theta(u)(a) + \theta(u)(b)$.

– $\theta(u)$ countably additive:
We will use a theorem from [42, §9 Theorem F], that a finitely additive finite measure μ is countably additive iff for all (non-strictly) decreasing sequences of measurable sets $(S_i)_{i \in \mathcal{N}}$ with empty intersection, $\lim_{i \to \infty} \mu(S_i) = 0$.
Therefore, we want to show that if $(a_i)_{i \in \mathcal{N}}$ is a sequence in A such that $a_j \leq a_i$ if $j \geq i$ and $\bigwedge\limits_{i=1}^{\infty} a_i = \bot$ then $\lim_{i \to \infty} \theta(u)(\llparenthesis a_i \rrparenthesis) = 0$. That is to say, we want to show that for all $\epsilon > 0$, there exists an $N \in \mathcal{N}$ such that for all $i \geq N$, $\theta(u)(\llparenthesis a_i \rrparenthesis) < \epsilon$. It suffices to prove that for all $\epsilon \in \mathbb{Q}_0$, with $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that for all $i \geq N$ $\theta(u)(\llparenthesis a_i \rrparenthesis) < \epsilon$. Observe

$$\begin{aligned}
\theta(u)(\llparenthesis a_i \rrparenthesis) < \epsilon &\Leftrightarrow \theta_{a_i}(u) < \epsilon \\
&\Leftrightarrow \neg(\theta_{a_i}(u) \geq \epsilon) \\
&\Leftrightarrow u \notin \theta_{a_i}^{-1}([\epsilon, 1]) \\
&\Leftrightarrow L_\epsilon(a_i) \notin u.
\end{aligned}$$

As the sequence (a_i) is descending, it suffices to show that there exists an $N \in \mathcal{N}$ such that $L_\epsilon(a_N) \notin u$ and it will hold for all $j \geq N$ by AA6 and up-closedness of ultrafilters.
So we have reduced showing countable additivity to showing that for all $\epsilon \in \mathbb{Q}_0$, $\epsilon > 0$, there exists an $N \in \mathcal{N}$ such that $L_\epsilon(a_N) \notin u$. By AA8, $\bigwedge_{i=1}^{\infty} a_i = \bot$ implies $\bigwedge_{i=1}^{\infty} L_\epsilon(a_i) = \bot$, so $\bigwedge_{i=1}^{\infty} L_\epsilon(a_i) \notin u$. If it were the case that $L_\epsilon(a_i) \in u$ for all $i \in \mathcal{N}$, the fact that u is a σ-ultrafilter would show that $\bigwedge_{i=1}^{\infty} L_\epsilon(a_i) \in u$, which would contradict its being an ultrafilter, so there must exist an $N \in \mathcal{N}$ such that $L_\epsilon(a_N) \notin u$.

- We want to show that for each S in the Giry σ-field on $\mathcal{G}(\mathcal{U}^\sigma(A))$, $\theta^{-1}(S) \in \mathcal{F}(A)$. As preimages preserve σ-Boolean operations, it suffices to show this for a set generating the Giry σ-field. As $\{[r, 1]\}_{r \in \mathbb{Q}_0}$ generates $\mathcal{B}o([0, 1])$, we have that

$$\{p_{\llparenthesis a \rrparenthesis}^{-1}([r, 1])\}_{a \in A, r \in \mathbb{Q}_0}$$

generates the Giry σ-field (again, using preservation of σ-Boolean operations under preimage).

We only need, therefore to show that $\theta^{-1}(p_{\llparenthesis a \rrparenthesis}^{-1}([r, 1])) \in \mathcal{F}(A)$. We reason as follows:

$$\begin{aligned}
\theta^{-1}(p_{\llparenthesis a \rrparenthesis}^{-1}([r, 1])) &= \{u \in \mathcal{U}^\sigma(A) \mid u \in \theta^{-1}(p_{\llparenthesis a \rrparenthesis}^{-1}([r, 1]))\} \\
&= \{u \in \mathcal{U}^\sigma(A) \mid p_{\llparenthesis a \rrparenthesis}(\theta(u)) \in [r, 1]\} \\
&= \{u \in \mathcal{U}^\sigma(A) \mid \theta(u)(\llparenthesis a \rrparenthesis) \in [r, 1]\} \\
&= \{u \in \mathcal{U}^\sigma(A) \mid \theta_a(u) \in [r, 1]\} \\
&= \theta_a^{-1}([r, 1]) \\
&= \llparenthesis L_r(a) \rrparenthesis \in \mathcal{F}(A).
\end{aligned}$$

The space $(A, \mathcal{F}(A))$ is σ-perfect by Theorem 5, so this is a σ-perfect Markov process. $\square$

Proposition 14. *With the above definition, G is a functor $\mathbf{Aumann}^{\mathrm{op}} \to \mathbf{PMarkov}$.*

Proof. By Proposition 13, it is defined correctly on objects. If we have a morphism of σ-Aumann algebras $(A, (L_r)_{r \in \mathbb{Q}_0}) \to (B, (M_r)_{r \in \mathbb{Q}_0})$, by Theorem 5, this defines a measurable map $G(f) : \mathcal{U}^\sigma(B) \to \mathcal{U}^\sigma(A)$, and the identity map and composition are preserved. Therefore we only need to show that $G(f)$ is a map of Markov processes, *i.e.* that the diagram:

$$\begin{tikzcd}
\mathcal{U}^\sigma(B) \arrow[r, "G(g)"] \arrow[d, "\lambda"'] & \mathcal{U}^\sigma(A) \arrow[d, "\theta"] \\
\mathcal{G}(\mathcal{U}^\sigma(B)) \arrow[r, "\mathcal{G}(G(g))"'] & \mathcal{G}(\mathcal{U}^\sigma(A))
\end{tikzcd}$$

commutes, where θ and λ are the morphisms defining the Markov processes on $\mathcal{U}^\sigma(A)$ and $\mathcal{U}^\sigma(B)$ respectively.

The bottom left path is

$$\mathcal{G}(G(g))(\lambda(u))(\llparenthesis a \rrparenthesis) = \lambda(u)(G(g)^{-1}(\llparenthesis a \rrparenthesis))$$

Now,

$$\begin{aligned}
G(g)^{-1}(\llparenthesis a \rrparenthesis) &= \{u \in \mathcal{U}^\sigma(B) \mid u \in G(g)^{-1}(\llparenthesis a \rrparenthesis)\} \\
&= \{u \in \mathcal{U}^\sigma(B) \mid G(g)(u) \in \llparenthesis a \rrparenthesis\} \\
&= \{u \in \mathcal{U}^\sigma(B) \mid a \in G(g)(u)\} \\
&= \{u \in \mathcal{U}^\sigma(B) \mid a \in g^{-1}(u)\} \\
&= \{u \in \mathcal{U}^\sigma(B) \mid g(a) \in u\} \\
&= \{u \in \mathcal{U}^\sigma(B) \mid u \in \llparenthesis g(a) \rrparenthesis\} \\
&= \llparenthesis g(a) \rrparenthesis,
\end{aligned}$$

so the bottom left path is equal to

$$\lambda(u)(\llparenthesis g(a) \rrparenthesis) = \lambda_{g(a)}(u). \tag{9}$$

The top right path is equal to:

$$\theta(G(g)(u))(\llparenthesis a \rrparenthesis) = \theta_a(G(g)(u)) \tag{10}$$

To show that the right hand sides of (9) and (10) are equal, we will show that $\lambda_{g(a)} = \theta_a \circ G(g)$ using Theorem 5. Let $r \in \mathbb{Q}_0$. Then

$$\begin{aligned}
\lambda^{-1}_{g(a)}([r,1]) &= \llparenthesis M_r(g(a)) \rrparenthesis && \text{definition of } \lambda_{g(a)} \\
&= \llparenthesis g(L_r(a)) \rrparenthesis && g \text{ a } \sigma\text{-AA morphism} \\
&= G(g)^{-1}(\llparenthesis L_r(a) \rrparenthesis) && \text{naturality (Theorem 5)} \\
&= G(g)^{-1}(\theta_a^{-1}([r,1])) && \text{definition of } \theta_a \\
&= (\theta_a \circ G(g))^{-1}([r,1]).
\end{aligned}$$

As intervals of the form $\{[r,1]\}_{r \in \mathbb{Q}_0}$ generate $\mathcal{B}o([0,1])$ (Lemma 11), we have

$$\lambda^{-1}_{g(a)}(S) = (\theta_a \circ G(g))^{-1}(S)$$

for all Borel subsets S of $[0,1]$. As $\lambda_{g(a)}$ and $\theta_a \circ G(g)$ are both maps $(\mathcal{U}^\sigma(B), \mathcal{F}(B)) \to ([0,1], \mathcal{B}o([0,1]))$, *i.e.* measurable maps between σ-perfect measurable spaces, we can apply the categorical duality from Theorem 5 to deduce $\lambda_{g(a)} = \theta_a \circ G(g)$ from the equation above, and therefore the diagram commutes. □

Theorem 15. *F is a left adjoint to G, and when restricted they define adjoint equivalences* $\mathbf{AumannSp}^{\mathrm{op}} \simeq \mathbf{PMarkov}$.

Proof. Recall the natural transformations $\llparenthesis\text{-}\rrparenthesis : A \to F(G(A))$ and $\langle\text{-}\rangle : (X,\Sigma) \to G(F(X,\Sigma))$ from Theorem 5. If we show that $\llparenthesis\text{-}\rrparenthesis$ is a morphism of Aumann algebras and $\langle\text{-}\rangle$ a morphism of Markov processes, then the commutativity of the naturality diagrams and the triangle diagrams defining an adjunction follows from the proofs in Theorem 5, and we have shown F is a left adjoint to G.

We first show that $\llparenthesis\text{-}\rrparenthesis$ is a σ-Aumann algebra morphism. That is to say, we want to show that for all $a \in A$ and $r \in \mathbb{Q}_0$ that $\llparenthesis L_r(a) \rrparenthesis = M_r(\llparenthesis a \rrparenthesis)$, where $(M_r)_{r \in \mathbb{Q}_0}$ is the Aumann algebra structure on $F(G(A))$. Well,

$$\begin{aligned}
L_r(\llparenthesis a \rrparenthesis) &= \{u \in \mathcal{U}^\sigma(A) \mid \theta(u)(\llparenthesis a \rrparenthesis) \geq r\} && \text{see (3)} \\
&= \{u \in \mathcal{U}^\sigma(A) \mid \theta_a(u) \geq r\} && \text{see (5)} \\
&= \theta_a^{-1}([r,1]) \\
&= \llparenthesis L_r(a) \rrparenthesis && \text{see (4).}
\end{aligned}$$

We now show that $\langle\text{-}\rangle$ is a morphism of Markov processes, *i.e.* the following diagram commutes

$$\begin{array}{ccc}
(X,\Sigma) & \xrightarrow{\langle\text{-}\rangle} & G(F(X,\Sigma)) \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\lambda} \\
\mathcal{G}(X,\Sigma) & \xrightarrow[\mathcal{G}(\langle\text{-}\rangle)]{} & \mathcal{G}(G(F(X,\Sigma))),
\end{array}$$

where λ is map making $G(F(X,\Sigma))$ a Markov process. In equations, what we want to show is that $\mathcal{G}(\langle\text{-}\rangle) \circ \theta = \lambda \circ \langle\text{-}\rangle$. Recall that the σ-field on $G(F(X,\Sigma))$ consists of elements of the form $\llparenthesis S \rrparenthesis$ for $S \in \Sigma$, so we want to show that, for all $x \in X$ and $S \in \Sigma$,

$$\mathcal{G}(\langle\text{-}\rangle)(\theta(x))(\llparenthesis S \rrparenthesis) = \lambda(\langle x \rangle)(\llparenthesis S \rrparenthesis). \tag{11}$$

If we expand the definition of $\mathcal{G}$ on the left hand side, we get

$$\mathcal{G}(\langle\text{-}\rangle)(\theta(x))(\llparenthesis S \rrparenthesis) = \theta(x)(\langle\text{-}\rangle^{-1}(\llparenthesis S \rrparenthesis)).$$

Now, we can simplify the argument of $\theta(x)$ as follows

$$\begin{aligned}
\langle\text{-}\rangle^{-1}(\llparenthesis S \rrparenthesis) &= \{x \in X \mid \langle x \rangle \in \llparenthesis S \rrparenthesis\} \\
&= \{x \in X \mid S \in \langle x \rangle\} \\
&= \{x \in X \mid x \in S\} = S,
\end{aligned}$$

so, all together, the left hand side of (11) is $\theta(x)(S)$. For the right hand side, we can expand the definition

$$\lambda(\langle x \rangle)(\llparenthesis S \rrparenthesis) = \lambda_S(\langle x \rangle)$$

according to (5). We now prove that $\lambda_S(\langle x \rangle) = \theta(x)(S)$ by showing that, for all $r \in \mathbb{Q}_0$, $\lambda_S(\langle x \rangle) \geq r$ iff $\theta(x)(S) \geq r$.

$$\begin{aligned}
\lambda_S(\langle x \rangle) \geq r &\Leftrightarrow \langle x \rangle \in \lambda_S^{-1}([r,1]) \\
&\Leftrightarrow \langle x \rangle \in \llparenthesis L_r(S) \rrparenthesis && (4) \\
&\Leftrightarrow L_r(S) \in \langle x \rangle \\
&\Leftrightarrow x \in L_r(S) \\
&\Leftrightarrow \theta(x)(S) \geq r && (3).
\end{aligned}$$

Because every real is the supremum of the rationals below it, this implies that $\theta(x)(S) = \lambda_S(\langle x \rangle)$, and therefore that (11) holds.

As we explained at the start of the proof, this suffices to show that F is a left adjoint to G. We can show that F and G define an adjoint equivalence $\mathbf{PMarkov} \simeq \mathbf{AumannSp}$ by showing that when $\llparenthesis\text{-}\rrparenthesis$ and $\langle\text{-}\rangle$ are, respectively, σ-Boolean algebra isomorphisms and measurable isomorphisms, they are σ-Aumann algebra isomorphisms and Markov process isomorphisms.

We do so as follows. We first need to show that $\llparenthesis\text{-}\rrparenthesis^{-1}$ is a σ-Aumann algebra homomorphism. First, we observe that the equality $M_r(\llparenthesis a \rrparenthesis) = \llparenthesis L_r(a) \rrparenthesis$ implies

$$\begin{aligned}
\llparenthesis\text{-}\rrparenthesis^{-1}(M_r(\llparenthesis a \rrparenthesis)) &= L_r(a) \\
&= L_r(\llparenthesis\text{-}\rrparenthesis^{-1}(\llparenthesis a \rrparenthesis)).
\end{aligned}$$

As every element of the algebra $F(G(A))$ is of the form $\llparenthesis a \rrparenthesis$ for some $a \in A$, we have shown that $\llparenthesis\text{-}\rrparenthesis^{-1}$ is an Aumann algebra isomorphism.

It is a generally true fact that a measurable isomorphism that is a map of Markov processes is an isomorphism of Markov processes, but we give the special case that $\langle\text{-}\rangle^{-1}$ is a morphism of Markov processes here:

$$\begin{aligned}
\mathcal{G}(\langle\text{-}\rangle) \circ \theta = \lambda \circ \langle\text{-}\rangle &\Leftrightarrow \mathcal{G}(\langle\text{-}\rangle) \circ \theta \circ \langle\text{-}\rangle^{-1} = \lambda \\
&\Leftrightarrow \theta \circ \langle\text{-}\rangle^{-1} = \mathcal{G}(\langle\text{-}\rangle^{-1}) \circ \lambda.
\end{aligned}$$

□