

# Categorical Duality in Probability and Quantum Foundations

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*Dedicated to my parents, Rosemary Johnston and James Furber.*

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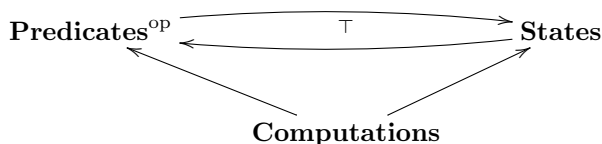


# Introduction

When we consider the semantics of programs, we can consider *state transformers* and *predicate transformers*. A *state transformer* describes the action of the program, taking its initial state to its final state. Predicate transformers go in the opposite direction and describe how to take a predicate to its *weakest precondition*. In the classical case of nondeterminism [96, 114] an isomorphism between state transformers and predicate transformers is obtained by restricting predicate transformers to those that are *healthy* [114].

This is the reason for our focus on categorical dualities, as the relationship between state and predicate transformers is best expressed as a contravariant equivalence of categories.

In general, as well as a category of predicates and a category of states, there is also a category of computations. Together, these form a *state-and-effect triangle* [56]. That is to say, we have functors between these categories as follows:



such that the diagram commutes up to isomorphism. In general, the definition of a state-and-effect triangle requires only that the two functors at the top form an adjunction [56], but we aim to get an equivalence. State-and-effect triangles appear in [52], and are considered in detail in [57].

In the first chapter, we concern ourselves with probabilistic computation. We prove that  $\mathcal{Kl}(\mathcal{R}) \simeq \mathbf{CC}^*\mathbf{Alg}_{\mathbf{PU}}^{\text{op}}$ , which can be considered to be a probabilistic generalization of Gelfand duality. We also give a brief introduction to effect modules, their relationship to order-unit spaces, and the expectation monad.

Chapters two to four can be considered to form, taken together, a non-commutative generalization of chapter one. In fact, we move to a setting that not only includes non-commutative  $C^*$ -algebras, but also situations in which a multiplication cannot even be defined. The convex set  $\mathcal{D}(X)$  of all finitely supported probability distributions can be considered to be the “state space” of some system, with elements of  $X$ , or their corresponding Dirac distributions, being the “pure states”, and  $\mathcal{D}(X)$  being obtained by forming “mixed states”, where convex combinations correspond to randomized mixtures. In quantum mechanics, one considers the convex set of density matrices  $\mathcal{DM}(\mathcal{H})$ , on a Hilbert space  $\mathcal{H}$ , as a state space. Again, the way of interpreting randomized probabilistic mixtures of states is by forming convex combinations. However, this time there is a difference – in  $\mathcal{D}(X)$  each mixed state is expressible *uniquely* as a convex combination of pure states, but this does not hold for  $\mathcal{DM}(\mathcal{H})$  when  $\dim \mathcal{H} \geq 2$ . The convex structure of  $\mathcal{DM}(\mathcal{H})$  can be considered to be the “probabilistic part” of quantum mechanics. One can then extend this idea to interpret convex sets, of various kinds, as state spaces in some generalized theory of probability. This is the viewpoint of *generalized probabilistic theories* [23, 31, 47, 91, 123, 9, 12, 11].

The reason we try to find a state-and-effect triangle for the category of non-commutative  $C^*$ -algebras is that quantum programs can always be represented using morphisms between non-commutative  $C^*$ -algebras. We might only use, for example, finite-dimensional  $C^*$ -algebras, or only those of the form  $B(\mathcal{H})$ , or only  $B(\mathbb{C}^{2^n})$ , but as long as we can make a state-and-effect triangle for  $C^*$ -algebras, we can restrict it to obtain one for these cases.

In the second chapter, we give an introduction to base-norm spaces and their relationship to convex sets. This is mainly for application in the next chapter. Base-norm spaces are intended as a way of “freely” producing an ordered vector space for a convex set to live inside. We have tried to clear up any confusion regarding inequivalent definitions of base-norm space that are in use by various authors, comparing several of these definitions to each other with explicit examples, and introducing the term pre-base-norm space for the weakest definition in common use. We give a proof of the known fact that every bounded convex set can be embedded as the base of a pre-base-norm space (making an equivalence  $\mathbf{PreBNS} \simeq \mathbf{BConv}$ ), and use this to slightly generalize a result from the literature to show that sequentially complete bounded convex sets are exactly those convex sets embeddable as bases of Banach base-norm spaces (so  $\mathbf{BBNS} \simeq \mathbf{CBConv}$ ). We then give an adjunction between base-norm and order-unit spaces, based on the duality between states and effects, and restrict this to an equivalence in the standard way. Then we briefly discuss how this equivalence is not quite adequate as a

generalization of the commutative case in chapter one.

In the third chapter, we first introduce a new characterization of Akbarov's *Smith spaces* and their duality with Banach spaces. From this we can produce two dualities involving base-norm and order-unit spaces, depending on whether one takes the base-norm spaces or the order-unit spaces to be Smith spaces. If we take the base-norm spaces to be Smith, we get a state-and-effect triangle where the category of computations is  $C^*$ -algebras, which gives us one possible state-transformer and predicate-transformer pair of semantics for quantum programs. In fact, the state-transformer semantics corresponds to what is called the *Schrödinger picture*, and the predicate transformer semantics to the *Heisenberg picture*.

We also produce another state-and-effect triangle, having the category of  $W^*$ -algebras and normal maps as computations, where the order-unit spaces are Smith. We show in each case (whether it is  $C^*$  or  $W^*$ -algebras) how to turn the base-norm and order-unit space equivalences into equivalences between a category of convex sets (**CBConv** or **CCL**) and a category of effect modules (**CEMod** and **BEMod**, respectively). We would prefer, in fact, to prove the dualities directly in this setting, but we could only do so using the extra facilities available in the vector space setting (linear independence, locally convex topologies, and the Hahn-Banach and bipolar theorems).

These dualities give two generalizations of the duality between states and effects in [24, §3.4] to the infinite-dimensional case, although with the drawback that we consider only positive maps and not completely positive ones. A similar generalization was considered before by Rennela [101, Theorem 4.1, Appendix C] (see also [17, Proposition 5.1]). Rennela's version was more order-theoretic, using a different characterization of normal maps of  $W^*$ -algebras, and with an adjunction between states and effects in the general case. This adjunction is not known to be an equivalence. This is a difficulty we were able to circumvent by using locally convex topologies.

In the third chapter we also use Smith order-unit spaces to show that **CBConv** is a reflective subcategory of two categories of Eilenberg-Moore algebras,  $\mathcal{EM}(\mathcal{D})$  and  $\mathcal{EM}(\mathcal{D}_\infty)$ , which are variations of a theorem [91, Theorem 3] proved by Ozawa that **BBNS** is a reflective subcategory of the category of preconvex structures. In chapter four we use a theorem of Świrszcz to show that **CCL** has two characterizations, as  $\mathcal{EM}(\mathcal{R})$  and  $\mathcal{EM}(\mathcal{E})$  (algebras of the Radon and expectation monads, respectively). We can then characterize **CEMod**, the compact effect modules, independently of an embedding in a topological vector space.

We now outline the original contributions. The probabilistic Gelfand duality in the first chapter is new, at least in the form presented there (using a

Kleisli category). We explain in section 1.6 how it is related to certain results on Markov kernels due to Umegaki. In the second chapter, the characterization of bases of base-norm spaces as sequentially complete bounded convex sets is new, as is the adjunction and equivalence for base-norm and order-unit spaces, in this categorical form (concerning morphisms as well as the spaces themselves). In chapter 3, the generalization of Akbarov's characterization of Smith spaces is new, and Smith base-norm and order-unit spaces are new definitions, so the equivalences given there are new, although the fact that it is possible to characterize dual spaces of base-norm and order-unit spaces using compactness in locally convex topologies is already known and relevant attributions are given in the text. The universal enveloping compact effect module described there is also new. In chapter four, the intrinsic definition of a compact effect module is new.

Chapter 1 was published as [44], except for the introductory parts on effect modules and the expectation monad, which come from [60], and Section 1.6, which is unpublished. In chapter 4, section 4.2 was published in [44], except for the different proof of Lemma 4.2.5, and section 4.3 was published in [60]. The rest of chapter 4 is unpublished, as are chapters 2 and 3.

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# Chapter 0

## Preliminaries

*The section on monads originated in [60], though it has been altered since then.*

There are a number of preliminaries we must get out of the way.

### 0.1 Convexity in Vector Spaces

Recall that a topological vector space is a vector space equipped with a topology such that addition and scalar multiplication are continuous with respect to the topology of the field over which the space is defined (in our case,  $\mathbb{R}$  or  $\mathbb{C}$ ) [110, I.1]

A subset  $S$  of a (real) vector space  $E$  is *convex* if for each  $x, y \in S$  and  $\alpha \in [0, 1]$ , we have  $\alpha x + (1 - \alpha)y \in S$ . An equivalent characterization is that for any finite sequence  $(\alpha_i)_{i \in I}$  of elements of  $[0, 1]$  such that  $\sum_{i \in I} \alpha_i = 1$ , and  $(x_i)_{i \in I}$  a finite sequence of elements of  $S$ , we have  $\sum_{i \in I} \alpha_i x_i \in S$ . The intersection of a family of convex subsets is convex, so convex sets form a lattice. The smallest convex set containing a subset  $S$  of a vector space  $E$  is called the *convex hull* and is written  $\text{co}(S)$ . It can be equivalently defined as the set of convex combinations of elements of  $S$ . A subset of a topological vector space is  *$\sigma$ -convex* if the analogous property for countable sequences  $(\alpha_i)_{i \in I}$  and  $(x_i)_{i \in I}$  holds, where the sum is interpreted as a limit of the sequence of finite sums in the usual manner.

We say a subset  $X$  of a vector space  $E$  is *absolutely convex* if for each finite set  $x_1, \dots, x_n$  in  $X$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  such that  $\sum_{i=1}^n |\alpha_i| \leq 1$  then  $\sum_{i=1}^n \alpha_i x_i \in X$ . The definition resembles that of convexity but with absolute values (and  $\leq 1$  instead of  $= 1$ ). We include the use of the empty set when

forming convex combinations, so every absolutely convex set contains 0. The effect of this is to rule out  $\emptyset$  as an absolutely convex subset of  $E$ , in spite of the fact that it is usually considered a convex subset of  $E$ . The astute reader will notice when this definition is required later on. There is a monadic theory of absolutely convex sets analogous to that for convex sets [99, 100], but as we do not require absolutely convex sets as independent objects we do not use it.

Unlike convexity, absolute convexity does not require an ordered field for definition, and so can also be defined for the complex numbers or even more general fields with valuation, such as  $p$ -adic fields. However, we will mostly use the definition over  $\mathbb{R}$ . For non-empty sets, absolute convexity is variously known as being *balanced and convex* [20, p. 102] or *circled and convex* [110, Chapter II, Exercise 1]. This is because a subset  $S$  of a real vector space  $E$  is *balanced* if  $x \in S$  implies  $-x \in S$ , and being balanced and convex is the same as being absolutely convex (Lemma A.3.1 in the appendix). We use the notation  $\text{absco}(X)$  to refer to the absolutely convex hull of  $X \subseteq E$ ,  $E$  a (real) vector space.

**Lemma 0.1.1.** *Let  $C$  be a non-empty subset of a vector space  $E$ . The absolutely convex hull of  $C$  is  $\text{co}(C \cup -C)$ .*

*Proof.*

- $\text{co}(C \cup -C) \subseteq \text{absco}(C)$ :

Suppose we have some element of  $x \in \text{co}(C \cup -C)$ , written as a convex combination

$$x = \alpha_1 x_1 + \cdots + \alpha_{k-1} x_{k-1} + \alpha_k (-x_k) + \cdots + \alpha_n (-x_n)$$

where  $1 \leq k \leq n$ , possibly reordering so elements of  $X$  occur for  $i < k$  and elements of  $-X$  for  $i \geq k$ . We can define  $\{\beta_i\}_{1 \leq i \leq n}$  by taking  $\beta_i = \alpha_i$  for  $1 \leq i < k$  and  $\beta_i = -\alpha_i$  for  $i \geq k$ . We have that  $|\beta_i| = \alpha_i$  and so

$$\sum_{i=1}^n |\beta_i| = \sum_{i=1}^n \alpha_i = 1,$$

which means that  $x$  can also be expressed as the absolutely convex combination

$$x = \sum_{i=1}^n \beta_i x_i,$$

showing  $x \in \text{absco}(C)$ .

- $\text{absco}(C) \subseteq \text{co}(C \cup -C)$ :

Suppose we have  $x \in \text{absco}(C)$ , expressed as an absolutely convex combination

$$x = \sum_{i=1}^n \alpha_i x_i,$$

allowing  $n = 0$ . We define  $\beta_i = |\alpha_i|$  and  $y_i = \text{sgn}(\alpha_i)x_i$ . There is still the problem that  $\sum_{i=1}^n \beta_i$  could be strictly less than 1. We can pick an element of  $y \in C$ , as it is non-empty, and define  $y_{n+1} = y$  and  $y_{n+2} = -y$ , and

$$\beta_{n+1} = \beta_{n+2} = \frac{1 - \sum_{i=1}^n \beta_i}{2}.$$

If  $n = 0$ , these are the only two values of  $\beta_i$  that are defined.

Then the convex combination

$$\begin{aligned} \sum_{i=1}^{n+2} \beta_i y_i &= \sum_{i=1}^n |\alpha_i| \text{sgn}(\alpha_i) x_i + \beta_{n+1} y - \beta_{n+2} y \\ &= \sum_{i=1}^n \alpha_i x_i + \beta_{n+1} y - \beta_{n+1} y = x, \end{aligned}$$

which proves that  $x \in \text{co}(C \cup -C)$ . □

We say a subset  $S$  of a vector space  $E$  is *radially bounded* if for each line  $L$  through the origin,  $S \cap L$  is bounded in  $L$ . Boundedness in  $L$  is defined as boundedness in  $\mathbb{R}$  via any linear isomorphism  $L \cong \mathbb{R}$ . We say  $S$  is *radially compact* if it is radially bounded and  $S \cap L$  is always closed in  $L$ , or equivalently by the Heine-Borel theorem, that  $S \cap L$  is compact.

**Lemma 0.1.2.** *An absolutely convex set  $U$  is radially bounded iff it contains no line through the origin.*

*Proof.* If  $L$  is a line through the origin contained in  $U$ ,  $L \cap U = L$  and is therefore unbounded, so  $U$  is not radially bounded.

For the other way, suppose  $U$  is radially unbounded, which is to say that there exists a line  $L$  such that  $L \cap U$  is unbounded. This means that given a linear isomorphism  $i : \mathbb{R} \xrightarrow{\sim} L$ , for each  $n \in \mathbb{N}$  there is an  $x \in \mathbb{R}$  such that  $i(x) \in L \cap U$  and  $|x| \geq n$ . Given such an  $x$ , by absolute convexity of  $U$  and  $L$ , we have  $-i(x) = i(-x) \in L \cap U$  also, and then  $i([-n, n]) \subseteq i([-x, x]) \subseteq L \cap U$  by convexity. As images preserve unions and  $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ , we have  $i(\mathbb{R}) = L \subseteq U$ . □

The main reason for considering absolutely convex sets is their relationship to seminorms. We first define a relation between subsets of a real vector space. We say  $U$  *absorbs*  $V$ , for  $U, V \subseteq E$ ,  $E$  a vector space, if there is some nonnegative real  $\alpha$  such that for all  $\alpha' \geq \alpha$ ,  $V \subseteq \alpha'U$ . Given a set  $U \subseteq E$ , we say it is *absorbent* or *absorbing*<sup>1</sup> if for all  $x \in E$ ,  $U$  absorbs  $x$ .

**Lemma 0.1.3.** *In a real vector space  $E$ , with subsets  $S, T$ :*

- (i) *If  $S$  and  $T$  are absorbent, then  $S \cap T$  is absorbent. Therefore absorbent sets are closed under finite intersection.*
- (ii) *If  $S \subseteq T$  and  $S$  is absorbent,  $T$  is absorbent.*
- (iii) *If  $S$  is a convex set containing 0, then  $S$  is absorbent iff for all  $x \in E$  there exists  $\lambda \in \mathbb{R}_{\geq 0}$  such that  $x \in \lambda S$ .*

Taken together, (i) and (ii) show that absorbent sets are a filter on  $E$ .

*Proof.*

- (i) Let  $x \in E$ . There exist  $\alpha, \beta \in \mathbb{R}_{>0}$  such that for all  $\alpha' \geq \alpha$ ,  $x \in \alpha'S$  and for all  $\beta' \geq \beta$ ,  $x \in \beta'T$ . Take  $\gamma = \max\{\alpha, \beta\} + 1$ , and observe that for all  $\gamma' \geq \gamma$ ,  $x \in \gamma'S$  and  $x \in \gamma'T$ , so  $x \in \gamma'S \cap \gamma'T$ . We also have that  $\gamma > 0$ , so  $\gamma \cdot -$  is a bijection, so  $\gamma \cdot -$  preserves Boolean operations. Therefore  $\gamma S \cap \gamma T = \gamma(S \cap T)$ , so  $S \cap T$  is absorbent.
- (ii) Let  $x \in E$ . There exists  $\alpha \in \mathbb{R}_{>0}$  such that for all  $\alpha' \geq \alpha$ ,  $x \in \alpha'S \subseteq \alpha'T$ , so  $T$  is absorbent.
- (iii) It is clear that absorbency of  $S$  implies that there exists such a  $\lambda$ , so we reduce to proving the other direction. Assume that for all  $x \in E$ , there exists  $\lambda \in \mathbb{R}_{\geq 0}$  such that  $x \in \lambda S$ . We show that for all  $\lambda' \geq \lambda$ ,  $x \in \lambda'S$ . If  $\lambda = 0$ , then  $\lambda S = \{0\}$  so  $x = 0$ , and therefore  $x \in \lambda'S$  for all  $\lambda \in \mathbb{R}_{\geq 0}$ . We now reduce to the case that  $\lambda > 0$ , and therefore  $\lambda' > 0$ . In this case, the fact that  $x \in \lambda S$  is equivalent to the existence of some  $y \in S$  such that  $x = \lambda y$ . As  $S$  is convex and contains 0, we have that  $\frac{\lambda}{\lambda'}y + (1 - \frac{\lambda}{\lambda'}) \cdot 0 \in S$ , and therefore  $\lambda'S \ni \lambda'(\frac{\lambda}{\lambda'}y + (1 - \frac{\lambda}{\lambda'}) \cdot 0) = \lambda y = x$ .  $\square$

**Lemma 0.1.4.** *In a topological vector space, every neighbourhood of 0 is absorbent.*

---

<sup>1</sup>Known as *radial* in [110].



*Proof.* Let  $E$  be a topological vector space,  $N \subseteq E$  a neighbourhood of 0, and  $x \in E$ . As scalar multiplication is continuous,  $\cdot \cdot x : \mathbb{R} \rightarrow E$  is continuous. Therefore there is an  $\epsilon > 0$  such that  $(\epsilon, \epsilon) \cdot x \subseteq N$ . We therefore have that  $\frac{\epsilon}{2}x \in N$ , and so  $x \in \frac{2}{\epsilon}N$ . As it is also true for any  $\epsilon' \leq \frac{\epsilon}{2}$  that  $\epsilon'x \in N$ , we have that for any  $\alpha' \geq \frac{2}{\epsilon}$ ,  $x \in \alpha'N$  (by taking  $\epsilon' = \frac{1}{\alpha'}$ ).  $\square$

For absorbent absolutely convex sets, we can define the *Minkowski functional* (or *gauge* [110, II.1.4, page 39]), as

$$\|x\|_U = \inf\{\lambda > 0 \mid x \in \lambda U\}.$$

This defines a seminorm for each absorbent absolutely convex set  $U$ .

**Lemma 0.1.5.** *The following are equivalent for an absorbent absolutely convex subset  $U$  of  $E$ :*

- (i)  $\|\cdot\|_U$  is a norm.
- (ii) The only linear subspace of  $U$  is  $\{0\}$ .
- (iii)  $U$  contains no line through the origin.
- (iv)  $U$  is radially bounded.

*Proof.*

- (i)  $\Rightarrow$  (ii): Let  $F \subseteq E$  be a linear subspace such that  $F \subseteq U$ . Then for all  $\alpha \in [0, \infty)$ ,  $F = \alpha F \subseteq \alpha U$ , so we have that for all  $x \in F$ ,  $\|x\|_U = 0$  and so  $x = 0$  by  $\|\cdot\|_U$  being a norm.
- (ii)  $\Rightarrow$  (iii): A line through the origin is a linear subspace not equal to  $\{0\}$ .
- (iii)  $\Rightarrow$  (i): We prove the contrapositive. Suppose  $\|\cdot\|_U$  is not a norm, *i.e.* that there exists  $x \in E$ ,  $x \neq 0$  such that  $\|x\|_U = 0$ . For any  $\alpha > 0$ , we have that  $x \in \alpha^{-1}U$  and so  $\alpha x \in U$ . By absolute convexity,  $\alpha x \in U$  for negative and zero values too, and so the line generated by  $x$  lies in  $U$ , and it is non-trivial because  $x \neq 0$ .
- (iii)  $\Leftrightarrow$  (iv): See Lemma 0.1.2.  $\square$

In fact, the Minkowski functional and open unit ball define isomorphisms between open absolutely convex neighbourhoods of 0 and continuous seminorms in any topological vector space [110, II.1.5 and 1.6].

If  $(E, \|\cdot\|)$  is a seminormed space, we define  $\text{Ball}(E)$ , or  $\text{Ball}(\|\cdot\|)$  to disambiguate if there is more than one seminorm present, as

$$\text{Ball}(E) = \{x \in E \mid \|x\| \leq 1\}.$$

In the case that the seminorm is defined as the Minkowski functional of some set, we can show the following.

**Lemma 0.1.6.** *Let  $E$  be a real vector space and  $U \subseteq E$  be an absolutely convex absorbent set. Then  $\|x\|_U \leq 1$  iff  $x \in \alpha U$  for all  $\alpha$  such that  $1 < \alpha < \infty$ . Equivalently*

$$\text{Ball}(E) = \bigcap_{1 < \alpha < \infty} \alpha U.$$

*Proof.* Let  $x \in E$ .

- $\|x\|_U \leq 1 \Rightarrow \forall \lambda > 1. x \in \lambda U$ :

If  $\|x\|_U \leq 1$ , this means that  $\inf\{\lambda > 0 \mid x \in \lambda U\} \leq 1$ . If  $\lambda > 1$ , then  $\inf\{\lambda > 0 \mid x \in \lambda U\} < \lambda$ , so  $\lambda$  is not a lower bound for  $\{\lambda' > 0 \mid x \in \lambda' U\}$ , so there exists some  $\lambda' > 0$  such that  $x \in \lambda' U$  and  $\lambda \not\leq \lambda'$ , i.e.  $\lambda > \lambda'$ . Therefore  $\lambda' U \subseteq \lambda U$ , so  $x \in \lambda U$ .

- $(\forall \lambda > 1. x \in \lambda U) \Rightarrow \|x\|_U \leq 1$ :

Suppose that  $x \in \lambda U$  for all  $\lambda > 1$ . Then we have

$$(1, \infty) \subseteq \{\lambda > 0 \mid x \in \lambda U\},$$

so

$$\|x\| = \inf\{\lambda > 0 \mid x \in \lambda U\} \leq \inf(1, \infty) = 1. \quad \square$$

Recall that a norm  $\|\cdot\|$  on a vector space  $E$  defines a metric  $d(x, y) = \|x - y\|$ , and this metric defines a topology on  $E$ , the  $\|\cdot\|$ -topology [110, II.2][20, III.1]. A Banach space is a normed space in which this metric is complete.

**Lemma 0.1.7.** *If  $E$  is a real vector space and  $U \subseteq E$  a radially compact absolutely convex absorbent set, the closed unit ball of  $\|\cdot\|_U$  is  $U$ .*

*Proof.* The closed unit ball is  $U' = \{x \in E \mid \forall \lambda \in (1, \infty). x \in \lambda U\}$  by Lemma 0.1.6. If  $x \in U$ , and  $\lambda$  is a real number  $> 1$ , then  $\lambda^{-1} \in (0, 1)$ , so  $\lambda^{-1} \cdot x \in U$ . Therefore  $\lambda \lambda^{-1} \cdot x \in \lambda U$  so  $x \in \lambda U$ , and hence  $x \in U'$ . This means  $U \subseteq U'$ .

Now suppose  $x \in U'$ . If  $x = 0$ , then  $x \in U$ , so we reduce to the case  $x \neq 0$ . Let  $L$  be the line generated by  $x$ . Consider the set  $M = \{\alpha^{-1}x \mid \alpha > 1\}$ .

Since  $x \in U'$ , we have that for all  $\alpha > 1$ ,  $\alpha^{-1}x \in U$ , and therefore  $M \subseteq U$ . By linearity,  $M \subseteq U \cap L$ . As  $U$  is radially compact,  $U \cap L$  is compact, and therefore closed, so the closure of  $M$  is also contained in  $U \cap L$ . Therefore we only need to show that  $x \in \overline{M}$ . We do this by showing that every neighbourhood of  $x$  intersects  $M$ .

Let  $\epsilon > 0$ , define  $\epsilon' = \max\{\epsilon, \frac{3}{2}\}$  and define

$$\alpha = \frac{1}{1 - \frac{\epsilon'}{2}}.$$

Since  $0 < 1 - \frac{\epsilon'}{2} < 1$ , we have  $\alpha > 1$ , as well as being defined. Now

$$\begin{aligned} \|x - \alpha^{-1}x\|_U &= \|(1 - \alpha^{-1})x\|_U = (1 - \alpha^{-1})\|x\|_U \\ &\leq (1 - \alpha^{-1}) \qquad \text{Since } x \in U', \text{ so } \|x\|_U = 1 \\ &= \frac{\epsilon'}{2} < \epsilon' \leq \epsilon. \end{aligned}$$

All together, this states  $\|x - \alpha^{-1}x\|_U < \epsilon$  for all  $\epsilon > 0$ . Since  $\alpha^{-1}x \in M$ , we have that  $x \in \overline{M}$  and  $x \in U$ , as required.  $\square$

If  $E, F$  are normed spaces, a map  $f : E \rightarrow F$  is *bounded* iff the following supremum exists

$$\|f\| = \sup\{\|f(x)\| \mid x \in E \text{ and } \|x\| \leq 1\}$$

Boundedness is equivalent to continuity for maps of normed spaces, and as indicated, the supremum above is a norm on continuous linear maps  $E \rightarrow F$  [20, Proposition III.2.1].

**Lemma 0.1.8.** *Let  $E, F$  be real vector spaces and  $U \subseteq E$ ,  $V \subseteq F$  be absolutely convex absorbent sets such that  $\|\cdot\|_U$  and  $\|\cdot\|_V$  are norms. If  $f : E \rightarrow F$  is a linear map such that  $f(U) \subseteq V$ , then  $\|f\| \leq 1$ . Consequently, if  $f(U) \subseteq \alpha V$ , where  $\alpha \in \mathbb{R}_{>0}$ , then  $\|f\| \leq \alpha$ , and so  $f$  is bounded.*

*Proof.* Let  $x \in X$  and  $\|x\|_U \leq 1$ . By Lemma 0.1.6 this is equivalent to  $x \in \alpha U$  for all  $\alpha > 1$ . We have by linearity of  $f$  that  $f(\alpha U) \subseteq \alpha V$ . Therefore  $f(x) \in \alpha V$  for all  $\alpha > 1$ , so by applying Lemma 0.1.6 in reverse, we conclude that  $\|f(x)\|_V \leq 1$ . As this applies for all  $x$  such that  $\|x\|_U \leq 1$ , we can conclude that  $\|f\| = \sup\{\|f(x)\|_V \mid x \in E, \|x\|_U \leq 1\} \leq 1$ .

If  $f(U) \subseteq \alpha V$ , then  $(\alpha^{-1}f)(U) \subseteq V$ , so  $\|\alpha^{-1}f\| \leq 1$ , so  $\|f\| \leq \alpha$ .  $\square$

We require the following definitions and lemmas about Banach spaces. We say that a family  $(x_i)_{i \in I}$  of elements of a Banach space  $E$  is *absolutely summable* if  $\sum_{i \in I} \|x_i\|$  converges. (See [95, §1.4] or [110, Chapter III Exercise 23 (iii)].) We use the notation  $\mathcal{P}_{\text{fin}}(X)$  for the *finite power set* of a set  $X$ , *i.e.* the set of finite subsets of  $X$ .

**Lemma 0.1.9.** *Let  $(x_i)_{i \in I}$  be a family of nonnegative reals such that  $\sum_{i \in I} x_i$  converges. Then the support of  $x_i$  is countable.*

*Proof.* The sum  $\sum_{i \in I} x_i$  is defined to be

$$\lim_{S \in \mathcal{P}_{\text{fin}}(I)} \sum_{i \in S} x_i,$$

and we will give the value of the sum the short name  $\lambda$ . Since the sum converges, we have that for all  $\epsilon > 0$  there exists  $S_\epsilon \in \mathcal{P}_{\text{fin}}(I)$  such that  $|\lambda - \sum_{i \in S_\epsilon} x_i| < \epsilon$ . The set  $S = \bigcup_{j=1}^{\infty} S_{2^{-j}}$  is a countable union of finite sets, hence countable. Suppose that there is some  $i' \in I \setminus S$  such that  $x_{i'} \neq 0$ . Then, taking a  $j$  such that  $x_{i'} > 2^{-j}$ , we have that

$$\lambda - \sum_{i \in S_j} x_i < 2^{-j} < x_{i'},$$

and so  $\sum_{i \in S_j} x_i + x_{i'} > \lambda$ . But since  $\lambda$  is the sum of  $(x_i)$  over all  $i \in I$  and  $i' \notin S_j$ , we have

$$\sum_{i \in S_j} x_i + x_{i'} \leq \lambda.$$

This contradicts our assumption that such an  $i'$  existed. Therefore the support of  $(x_i)_{i \in I}$  is contained in  $S$ , and hence is countable.  $\square$

**Corollary 0.1.10.** *Let  $(x_i)_{i \in I}$  be an absolutely summable family of elements of some normed space. The support of  $(x_i)$  is countable.*

*Proof.* Since  $(x_i)_{i \in I}$  is absolutely summable,  $(\|x_i\|)_{i \in I}$  is a summable sequence of nonnegative reals, and so has countable support by Lemma 0.1.9. Since  $\|x_i\| = 0$  iff  $x_i = 0$ ,  $(x_i)_{i \in I}$  has the same, countable, support as  $(\|x_i\|)_{i \in I}$ .  $\square$

The above applies in particular to sums of real numbers, as  $\mathbb{R}$  is a normed space. The corollary above can also be found as an exercise in [110, Chapter III Exercise 23 (c)]. Recall that a *poset* is a partially-ordered set, *i.e.* a set equipped with a reflexive, antisymmetric, transitive relation, and that  $\mathcal{P}_{\text{fin}}(X)$ , for any set  $X$ , is a poset under the usual ordering by  $\subseteq$ .

**Lemma 0.1.11.** *In any Banach space  $E$ , every absolutely summable family  $(x_i)_{i \in I}$  is summable, i.e.  $\sum_{i \in I} x_i$  converges.*

*Proof.* Let  $J = \mathcal{P}_{\text{fin}}(I)$ , being a directed poset under inclusion, and  $(S_j)_{j \in J}$  be the net defined as

$$S_j = \sum_{i \in j} x_i.$$

By definition,  $\sum_{i \in I} x_i$  converges iff  $S_j$  converges. We show  $S_j$  converges by showing that it is Cauchy, i.e. that for all  $\epsilon > 0$  there is a  $N_\epsilon \in J$  such that for all  $j, k \geq N_\epsilon$ ,  $\|S_j - S_k\| < \epsilon$ .

Let  $\epsilon > 0$ . We will consider the sum  $\sum_{i \in I} \|x_i\|$ . Define  $(S'_j)_{j \in J}$  as

$$S'_j = \sum_{i \in j} \|x_i\|$$

Since  $\sum_{i \in I} \|x_i\|$  converges, there is some  $N_\epsilon \in J$  such that for all  $j, k \geq N_\epsilon$ ,  $|S'_j - S'_k| < \epsilon$ .  $\square$

The previous lemma is found as an exercise in [110, Chapter III Exercise 23 (a)].

**Lemma 0.1.12.** *Let  $(x_i)_{i \in I}$  and  $(y_i)_{i \in I}$  be absolutely summable families in a Banach space  $E$ , with the same index set  $I$ . Then  $(x_i + y_i)_{i \in I}$  is absolutely summable and*

$$\sum_{i \in I} (x_i + y_i) = \sum_{i \in I} x_i + \sum_{i \in I} y_i.$$

*Proof.* By definition,

$$\sum_{i \in I} \|x_i + y_i\| = \lim_{S \in \mathcal{P}_{\text{fin}}(I)} \sum_{i \in S} \|x_i + y_i\|.$$

Since each term of the sum is non-negative, the net  $(\sum_{i \in S} \|x_i + y_i\|)_{S \in \mathcal{P}_{\text{fin}}(I)}$  is monotone. If we observe that for all  $S \in \mathcal{P}_{\text{fin}}(I)$

$$\sum_{i \in S} \|x_i + y_i\| \leq \sum_{i \in S} \|x_i\| + \|y_i\|$$

and that  $\lim_{S \in \mathcal{P}_{\text{fin}}(I)} \sum_{i \in S} \|x_i\| + \|y_i\|$  exists by continuity of addition (for real numbers), we can use Lemma A.1.2 to conclude that  $\sum_{i \in I} \|x_i + y_i\|$  converges. Therefore  $\sum_{i \in I} (x_i + y_i)$  converges by Lemma 0.1.11, and so

$$\sum_{i \in I} (x_i + y_i) = \sum_{i \in I} x_i + \sum_{i \in I} y_i$$

by continuity of addition in  $E$ . □

A *locally convex space* is a Hausdorff topological vector space where the convex neighbourhoods of each point form a neighbourhood base of that point, equivalently that the absolutely convex neighbourhoods of 0 form a neighbourhood base for 0 [110, II.4]. It is equivalent to require that the absolutely convex neighbourhoods of 0 form a neighbourhood base for 0 [110, I.1.2]. Under the correspondence between absorbing absolutely convex sets and seminorms, locally convex spaces are also exactly those spaces whose topology can be defined by a separating family of seminorms, and this is sometimes used as a definition [20, Definition IV.1.2]. Products of locally convex spaces are given by the topological product, and neighbourhoods of zero of the form  $U \times V$  for  $U$  a neighbourhood of zero in the left factor and  $V$  one in the right factor form a base [110, II.5.2 Products].

If  $E$  is a locally convex space, a subset  $S \subseteq E$  is said to be *bounded* if  $S$  is absorbed by all neighbourhoods of zero. If  $f : E \rightarrow F$  is a continuous linear map, then  $f(S)$  is a bounded subset of  $F$  [110, I.5.4].

We include here some basic lemmas about bounded sets in locally convex spaces.

**Lemma 0.1.13.**

- (i) *A set  $S \subseteq E$  is bounded in a topological vector space  $E$  iff it is absorbed by all elements of a neighbourhood base for 0.*
- (ii) *If  $X \subseteq E$ ,  $Y \subseteq F$  are bounded subsets of locally convex spaces  $E$  and  $F$ , then  $X \times Y$  is a bounded subset of  $E \times F$ .*
- (iii) *If  $X \subseteq E$  is a bounded subset,  $x \in E$ , then  $X + x$  is bounded.*

*Proof.*

- (i) The only if direction is clear. We therefore show that if  $S$  is absorbed by all the elements of a neighbourhood base  $\mathcal{N}$  for 0 in  $E$ ,  $S$  is bounded. Let  $U$  be a neighbourhood of 0, and  $N \in \mathcal{N}$  a basic neighbourhood such that  $N \subseteq U$ , which must exist by  $\mathcal{N}$  being a neighbourhood base. By assumption,  $N$  absorbs  $S$ , so there is an  $\alpha > 0$  such that  $S \subseteq \alpha N$ . Since  $\alpha N \subseteq \alpha U$ , we have that  $U$  absorbs  $S$ . Since this applies for an arbitrary neighbourhood of 0,  $S$  is bounded.
- (ii) By part (i), it suffices to show that if  $U \subseteq E$  and  $V \subseteq F$  are 0-neighbourhoods in  $E$  and  $F$  respectively, that  $U \times V$  absorbs  $X \times Y$ .

We know there exist  $\alpha, \beta > 0$  such that  $X \subseteq \alpha U$  and  $Y \subseteq \beta V$ . Let  $\gamma = \max\{\alpha, \beta\}$ . Then we have  $X \subseteq \gamma U$  and  $Y \subseteq \gamma V$ , so  $X \times Y \subseteq \gamma(U \times V)$ , as required.

- (iii) By part (i), it suffices to show that for any absolutely convex neighbourhood  $U$  of 0, there is an  $\alpha > 0$  such that  $X+x \subseteq \alpha U$ . Since  $X$  is bounded, there is a  $\beta > 0$  such that  $X \subseteq \beta U$ , and as  $U$  is absorbent, there is a  $\gamma > 0$  such that  $x \subseteq \gamma U$ . We can take  $\alpha = \beta + \gamma$ , i.e.  $X+x \subseteq (\beta + \gamma)U$ , because if  $y \in X$ , so  $\beta^{-1}y, \gamma^{-1}x \in U$  so

$$\begin{aligned} \frac{y+x}{\beta+\gamma} &= \frac{y}{\beta+\gamma} + \frac{x}{\beta+\gamma} \\ &= \frac{\beta}{\beta+\gamma}\beta^{-1}y + \frac{\gamma}{\beta+\gamma}\gamma^{-1}x \in U \end{aligned}$$

by convexity of  $U$ . □

**Lemma 0.1.14.** *Every compact subset of a locally convex space is bounded.*

*Proof.* Let  $E$  be a locally convex space, with  $K$  a compact subset and  $U$  a 0-neighbourhood, and  $V = \text{int}(U)$ , which is necessarily an open 0-nbhd. By Lemma 0.1.4,  $V$  is absorbent, so  $\{\alpha V\}_{\alpha \in \mathbb{R}_{>0}}$  is an open cover of  $E$ , and therefore of  $K$ . Applying compactness, we take a finite subcover, if we take the largest  $\beta \in \mathbb{R}_{>0}$  such that  $\beta V$  is in this subcover, it contains all the other sets in the subcover so  $K \subseteq \beta V$ . Therefore  $K \subseteq U$ , and so  $K$  is bounded. □

**Lemma 0.1.15.** *If  $S \subseteq E$  is bounded,  $E$  being a locally convex space, then its absolutely convex hull  $\text{absco}(S)$  is also bounded.*

*Proof.* Using Lemma 0.1.13 and local convexity of  $E$ , we only need to show that  $\text{absco}(S)$  is absorbed by all absolutely convex neighbourhoods of 0. So let  $U$  be an absolutely convex 0-neighbourhood. We know that there is an  $\alpha > 0$  such that  $S \subseteq \alpha U$ . We therefore have that  $\text{absco}(S) \subseteq \text{absco}(\alpha U) = \alpha U$  as  $\alpha U$  was absolutely convex to start off with. Therefore  $U$  absorbs  $\text{absco}(S)$  and so  $\text{absco}(S)$  is bounded. □

**Lemma 0.1.16.** *If  $X \subseteq E$  is bounded, for  $E$  locally convex, then  $X$  is radially bounded.*

*Proof.* Suppose for a contradiction that  $X$  is bounded, but radially unbounded. By Lemma 0.1.15,  $\text{absco}(X)$  is bounded, but also radially unbounded as it contains  $X$ . By Lemma 0.1.2 there exists a line through the origin in  $\text{absco}(X)$ , which we take to be generated by a nonzero element  $x \in \text{absco}(X)$ . The

boundedness of  $\text{absco}(X)$  implies that for each 0-neighbourhoods  $U$ , there is an  $\alpha > 0$  such that  $\text{absco}(X) \subseteq \alpha U$ , and so  $\beta x \in \text{absco}(X) \subseteq \alpha U$  for all  $\beta \in \mathbb{R}$ . This implies that  $x \in \frac{\alpha}{\beta} U$  for all  $\beta > 0$ , so by taking  $\beta = \alpha^{-1}$  we obtain  $x \in U$  for any 0-neighbourhood  $U$ .

Since  $E$  is Hausdorff, there are open sets  $U, V \subseteq E$  such that  $0 \in U$ ,  $x \in V$  and  $U \cap V = \emptyset$ . Therefore  $U$  does not contain  $x$ , contradicting the previous paragraph. We therefore have that  $X$  is radially bounded by contradiction.  $\square$

We also have to following lemma about products of locally convex spaces.

**Lemma 0.1.17.** *Let  $E \times F$  be a product of locally convex spaces. We have a map  $\kappa_1 : E \rightarrow E \times F$  defined as*

$$\kappa_1(x) = (x, 0).$$

*This is a continuous linear section of  $\pi_1$ , and hence a linear homeomorphism onto its image  $E \times 0$ . The analogous statements are true for  $\kappa_2$  and  $\pi_2$ .*

*Proof.* We only give the proof for  $\kappa_1$  and  $\pi_1$  as the other side is analogous. We see that  $\kappa_1$  is linear because addition and scalar multiplication in  $E \times F$  are pointwise. If  $U \times V$  is a basic neighbourhood of 0 in  $E \times F$ , then  $\kappa_1^{-1}(U \times V) = U$ , so  $\kappa_1$  is continuous. We can see that it is a section of  $\pi_1$  because

$$\pi_1 \circ \kappa_1(x) = \pi_1(x, 0) = x$$

for all  $x \in E$ . This implies that it is a homeomorphism onto its image in  $E \times F$ .  $\square$

We now deal with some notions relating to completeness in locally convex spaces. In any topological vector space  $E$ , we can define a uniformity<sup>2</sup> on  $E$  by taking a base of entourages to be the family of sets of the form

$$N_V = \{(x, y) \in E \times E \mid x - y \in V\}$$

where  $V$  runs through some base of 0-neighbourhoods [110, §I.1.4]. The topology defined by this uniformity is the topology  $E$  started with, and whenever we apply a notion relating to uniform spaces to topological vector spaces, we mean to use this uniformity. Any continuous linear map is uniformly continuous. A topological vector space is said to be complete iff it is complete in that uniformity.

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<sup>2</sup>See [16, §II.1.1] for the basic theory of uniform spaces.



**Lemma 0.1.18.** *For any locally convex space  $E$ ,  $a$  an element of  $E$ , the map  $- + a : E \rightarrow E$  is an affine uniform isomorphism.*

*Proof.* We first prove that  $- + a$  is affine and uniformly continuous. Consider a convex combination  $\alpha x + (1 - \alpha)y$  in  $E$  in the following:

$$\begin{aligned} (\alpha x + (1 - \alpha)y) + a &= \alpha x + (1 - \alpha)y + \alpha a + (1 - \alpha)a \\ &= \alpha(x + a) + (1 - \alpha)(y + a). \end{aligned}$$

To show it is uniformly continuous, let  $N_V$  be a basic entourage coming from a 0-neighbourhood  $V$ . We will show that  $N_V \subseteq ((- + a) \times (- + a))^{-1}(N_V)$ , equivalently  $((- + a) \times (- + a))(N_V) \subseteq N_V$  as follows. Let  $(x, y) \in N_V$ , i.e.  $x - y \in V$ . Then

$$((- + a) \times (- + a))(x, y) = (x + a) - (y + a) = x - y \in V$$

so  $(x + a, y + a) \in N_V$ .

We then observe that  $- + (-a)$  is of the same form, hence affine and uniformly continuous, and the inverse to  $- + a$ , so  $- + a$  is a uniform isomorphism.  $\square$

A sequence  $(x_i)_{i \in \mathbb{N}}$  in  $X$  is a *Cauchy sequence* if for each entourage  $U \subseteq X \times X$ , there is an  $N \in \mathbb{N}$  such that for all  $i, j \geq N$  we have  $(x_i, x_j) \in U$ . So in a topological vector space, a sequence is Cauchy if for each 0-neighbourhood (equivalently, for each basic 0-neighbourhood for some 0-neighbourhood base)  $U$ , there is an  $N \in \mathbb{N}$  such that for all  $i, j \geq N$  we have  $x_i - x_j \in U$ . If we consider a normed space as a locally convex space, with its 0-neighbourhood base of open balls of radius  $\epsilon$ , we see that this coincides with the usual notion of Cauchy sequence in normed spaces. A subset  $S$  of a locally convex space  $E$  is *sequentially complete* if every Cauchy sequence with values in  $S$  converges to a point in  $S$ .

**Lemma 0.1.19.** *Let  $(\alpha_i)_{i \in \mathbb{N}}$  be a sequence of real numbers,  $\alpha_i \geq 0$  for all  $i$ , such that  $\sum_{i=1}^{\infty} \alpha_i = 1$ . Let  $(x_i)_{i \in \mathbb{N}}$  be a sequence in a locally convex space  $E$  that is bounded (as a subset of  $E$ ). Then the sequence*

$$\left( \sum_{i=1}^n \alpha_i x_i \right)_{n \in \mathbb{N}}$$

*is Cauchy.*

*Proof.* Let  $U \subseteq E$  be an absolutely convex neighbourhood of 0. Since  $(x_i)_{i \in \mathbb{N}}$  is bounded, there is a  $\beta > 0$  such that  $x_i \in \beta U$ , or equivalently  $\frac{1}{\beta}x_i \in U$  for all  $i \in \mathbb{N}$ . Since  $(\sum_{i=1}^n \alpha_i)_{n \in \mathbb{N}}$  converges, as the sum of that series is 1, it is a Cauchy sequence, so there is an  $N \in \mathbb{N}$  such that for all  $m, n \geq N$  (without loss of generality taking  $m \geq n$ ) we have

$$\left| \sum_{i=n+1}^m \alpha_i \right| < \frac{1}{\beta},$$

and since each term of the sum is nonnegative

$$0 \leq \sum_{i=n+1}^m \alpha_i < \frac{1}{\beta},$$

and in fact

$$0 \leq \sum_{i=n+1}^m \beta \alpha_i < 1.$$

We can now see that for all  $n, m \geq N$ , without loss of generality taking  $m \geq n$ , we have

$$\sum_{i=1}^m \alpha_i x_i - \sum_{i=1}^n \alpha_i x_i = \sum_{i=n+1}^m \alpha_i x_i = \sum_{i=n}^m (\beta \alpha_i) \left( \frac{1}{\beta} x_i \right).$$

As this is an absolutely convex combination of elements of  $U$ , we have shown that  $\sum_{i=1}^m \alpha_i x_i \in U$ , as is required to show that the sequence is Cauchy.  $\square$

## 0.2 Ordered Vector Spaces

A *wedge* in a (real) vector space  $E$  is a subset  $E_+$  such that:

- (i) If  $x, y \in E_+$ ,  $x + y \in E_+$ .
- (ii) If  $\alpha \in \mathbb{R}$ ,  $\alpha \geq 0$ , and  $x \in E_+$ , then  $\alpha x \in E_+$ .

The wedge defines a pre-order on  $E$  by defining

$$x \leq y \Leftrightarrow y - x \in E_+. \quad (0.1)$$

In fact, this defines a map from vector spaces with a wedge to vector spaces that are also pre-ordered sets where the pre-order is translation invariant.

Taking the wedge of elements greater than or equal to zero defines the inverse map, so these two structures are equivalent.

We say that a wedge is a *cone* if  $E_+ \cap -E_+ = \{0\}$ . Some authors call this a *proper cone* and use *cone*, or even *convex cone* to mean a wedge, reserving *cone* for an even more general notion. We will stick to the previous terminology. For a cone, (0.1) defines a partial order. We will now use *partially ordered vector space* to refer to a pair  $(E, E_+)$ . In fact, we often omit the word *partially* and refer to these simply as *ordered vector spaces*. A linear map  $f : E \rightarrow F$  between partially ordered vector spaces  $(E, E_+)$  and  $(F, F_+)$  is *positive* if  $f(E_+) \subseteq F_+$ . For linear maps, this is equivalent to being monotone in the order (0.1). Partially ordered vector spaces and linear maps form a category.

Recall that a poset  $P$  is called *directed* if it is nonempty and each pair has an upper bound, *i.e.* if for each pair  $x, y \in P$  there exists  $z \in P$  such that  $x \leq z$  and  $y \leq z$ . We say a cone  $E_+ \subseteq E$  is *generating* if  $E$  is the (real) span of  $E_+$ , equivalently  $E_+ - E_+ = E$ . This is equivalent to the statement that each  $x \in E$  can be (nonuniquely, in general) expressed as  $x_+ - x_-$  with  $x_+, x_- \in E_+$ . Many authors say instead that  $(E, E_+)$  is directed, for the following reason.

**Proposition 0.2.1.** *A partially ordered vector space  $(E, E_+)$  is directed iff  $E_+$  is generating.*

*Proof.*

- Directed implies generating:

Let  $x \in E$ . Since  $(E, E_+)$  is directed, there exists an element, which we shall call  $x_+$ , such that  $x \leq x_+$  and  $0 \leq x_+$ . Applying (0.1), we see that  $x_+ \in E_+$  and  $x_+ - x \in E_+$ . If we define  $x_- = x_+ - x$ , we see that  $x = x_+ - x_-$  and  $x_+, x_- \in E_+$ , as required.

- Generating implies directed:

Let  $x, y \in E$ . We can choose decompositions of them into positive elements as  $x_+ - x_- = x$  and  $y_+ - y_- = y$ . These equations can also be written as

$$x_+ - x = x_- \qquad y_+ - y = y_-,$$

which by (0.1) implies  $x \leq x_+$  and  $y \leq y_+$ . Using the fact that  $x_+, y_+ \in E_+$  and the translation invariance of the order, we also have

$$x_+ \leq x_+ + y_+ \qquad y_+ \leq x_+ + y_+,$$

so we can apply transitivity of  $\leq$  to deduce

$$x \leq x_+ + y_+ \qquad y \leq x_+ + y_+.$$

We can therefore see that  $x_+ + y_+$  is an upper bound for  $\{x, y\}$ . Since  $x$  and  $y$  are arbitrary, and  $E$  can never be empty,  $E$  is directed.  $\square$

From now on we will use the common terminology and refer to  $(E, E_+)$  as directed if  $E_+$  is generating. We remark at this point that  $(\mathbb{R}, [0, \infty))$  is a directed ordered vector space and the order is the usual one.

We can extend the notation for closed intervals, as used on  $\mathbb{R}$ , to any ordered vector space. If  $(E, E_+)$  is an ordered vector space and  $a, b \in E$  is any pair of elements, we define

$$\begin{aligned} [a, b] &= \{x \in E \mid x - a \in E_+ \text{ and } b - x \in E_+\} \\ [a, \infty) &= \{x \in E \mid x - a \in E_+\} = E_+ + a \\ (-\infty, b] &= \{x \in E \mid b - x \in E_+\} = b - E_+. \end{aligned}$$

It is clear from these definitions that  $[a, b] = [a, \infty) \cap (-\infty, b] = a + E_+ \cap b - E_+$ .

**Lemma 0.2.2.** *Let  $(E, E_+)$  be an ordered vector space,  $a, b$  elements of  $E$ .*

(i) *If  $\alpha \in \mathbb{R}_{>0}$ . Then*

$$\alpha[a, b] = [\alpha a, \alpha b]$$

(ii) *If  $c \in E$ , then*

$$c + [a, b] = [a + c, b + c]$$

*Proof.*

(i) We reason as follows:

$$\begin{aligned} x \in \alpha[a, b] &\Leftrightarrow \alpha^{-1}x \in [a, b] \\ &\Leftrightarrow \alpha^{-1}x - a \in E_+ \text{ and } b - \alpha^{-1}x \in E_+ \\ &\Leftrightarrow x - \alpha a \in E_+ \text{ and } \alpha b - x \in E_+ \qquad E_+ \text{ a cone} \\ &\Leftrightarrow x \in [-\alpha a, \alpha b]. \end{aligned}$$

(ii) In this case:

$$\begin{aligned} x \in c + [a, b] &\Leftrightarrow x - c \in [a, b] \\ &\Leftrightarrow a \leq x - c \leq b \\ &\Leftrightarrow x - c - a \in E_+ \text{ and } b - x + c \in E_+ \\ &\Leftrightarrow x - (a + c) \in E_+ \text{ and } (b + c) - x \in E_+ \\ &\Leftrightarrow x \in [a + c, b + c] \end{aligned}$$

so the two sets are the same.  $\square$

### 0.3 Dualities, Polars and Bipolars

A *duality* is a triple  $(E, F, \langle -, - \rangle)$  where  $E, F$  are real vector spaces, and  $\langle -, - \rangle : E \times F \rightarrow \mathbb{R}$  is a bilinear map such that

$$\begin{aligned}\forall y \in F. \langle x, y \rangle = 0 &\text{ implies } x = 0 \\ \forall x \in E. \langle x, y \rangle = 0 &\text{ implies } y = 0.\end{aligned}$$

Some authors use *separated duality* to describe this, leaving the term duality to refer to a pair of vector spaces  $E, F$  with a bilinear map  $E \times F \rightarrow \mathbb{R}$ . The basic theory is described in [110, §IV.1] and [15, §II.6.1]. By the symmetry in the definition, if  $(E, F, \langle -, - \rangle)$  is a duality,  $(F, E, \langle -, - \rangle \circ \sigma_{E, F})$  is a duality, where  $\sigma_{E, F}(x, y) = (y, x)$ . We call this the *transpose* of a duality.

For any locally convex topological vector space  $E$ , we denote by  $E^*$  the vector space of  $k$ -valued continuous linear maps, the *dual space*, where  $k \in \{\mathbb{R}, \mathbb{C}\}$  is the base field of  $E$ . This is also used by some authors for the “algebraic dual”, of all linear maps, including discontinuous ones, who use  $E'$  for the continuous dual. However,  $E'$  is used to refer to the commutant in the theory of operator algebras, so we do not use it.

**Proposition 0.3.1.** *If  $E$  is a locally convex space, and we define*

$$\begin{aligned}\langle -, - \rangle &: E \times E^* \rightarrow \mathbb{R} \\ \langle x, \phi \rangle &= \phi(x)\end{aligned}$$

*then  $(E, E^*, \langle -, - \rangle)$  is a duality.*

*Proof.* See [110, §IV.1 Example 2] or [15, p. II.42].  $\square$

We can define a locally convex topology  $\sigma(E, F)$  on  $E$  with the following subbase of zero-neighbourhoods

$$N_{y, \epsilon} = \{x \in E \mid |\langle x, y \rangle| < \epsilon\}, \quad (0.2)$$

where  $y \in F$  and  $\epsilon \in \mathbb{R}_{>0}$ . In fact,  $(N_y) = (N_{y,1})_{y \in F}$  defines the same topology. This topology is the coarsest topology such that each  $\langle -, y \rangle : E \rightarrow \mathbb{R}$  is continuous. By transposing the duality, we can also define the topology  $\sigma(F, E)$  on  $F$ . In the special case of the duality from Proposition 0.3.1,  $\sigma(E, E^*)$  is called the *weak topology* and  $\sigma(E^*, E)$  the *weak- $^*$  topology*. If  $(E, F, \langle -, - \rangle)$  is a

duality, and  $Y \subseteq F$  is a set whose span is  $F$ , then we can define  $\sigma(E, Y)$  to be the topology with sets of the form  $N_{y, \epsilon}$  with  $y \in Y$  as a base. This topology agrees with  $\sigma(E, F)$ .

We will need the following fundamental results about linear maps that are continuous in weak topologies.

**Proposition 0.3.2.** *Let  $(E, F, \langle -, - \rangle)$  be a duality. The map  $x \mapsto \langle x, - \rangle$  defines a linear isomorphism from  $E$  to  $(F, \sigma(F, E))^*$ , i.e. from  $E$  to linear maps from  $F$  to  $\mathbb{R}$  that are continuous in the  $\sigma(F, E)$ -topology. By symmetry, the map  $y \mapsto \langle -, y \rangle$  defines a linear isomorphism from  $F$  to  $(E, \sigma(E, F))^*$ .*

*Proof.* See [110, IV.1.2]. □

**Proposition 0.3.3.** *Let  $(E_1, F_1, \langle -, - \rangle_1)$  and  $(E_2, F_2, \langle -, - \rangle_2)$  be dualities. Let  $f : E_1 \rightarrow E_2$  be a linear map. The following statements are equivalent:*

- (i)  *$f$  is continuous from  $\sigma(E_1, F_1)$  to  $\sigma(E_2, F_2)$ .*
- (ii) *There exists a linear map  $g : F_2 \rightarrow F_1$  such that for all  $x \in E_1$  and  $y \in F_2$*

$$\langle f(x), y \rangle_2 = \langle x, g(y) \rangle_1 \quad (0.3)$$

*Any  $g$  satisfying (0.3) is necessarily continuous from  $\sigma(F_2, E_2)$  to  $\sigma(F_1, E_1)$ .*

*Proof.* See [110, IV.2.1]. □

**Proposition 0.3.4.** *Let  $E$  be a locally convex space, and  $C$  a convex subset. Then  $C$  is closed iff it is  $\sigma(E, E^*)$ -closed, and the closure of  $C$  is the  $\sigma(E, E^*)$ -closure.*

*Proof.* See [110, II.9.2 Corollary 2]. □

### 0.3.1 Polars

Given a duality  $(E, F, \langle -, - \rangle)$ , and a subset  $S \subseteq E$ , we define the *polar* [110, §IV.1.3] of  $S$ ,  $S^\circ \subseteq F$ , to be

$$S^\circ = \{y \in F \mid \forall x \in S. \langle x, y \rangle \leq 1\}.$$

If  $T \subseteq F$ , we define  $T^\circ$  to be the polar of  $T$  in the transposed duality.

In [15, §II.6.3] the polar is defined to be

$$\{y \in F \mid \forall x \in S. \langle x, y \rangle \geq -1\}.$$

It is clear that this is  $-S^\circ$ , and so any results about polars from [15] can be translated to match the usual definitions.

We define the *absolute polar* as

$$S^{|\circ|} = \{y \in F \mid \forall x \in S. |\langle x, y \rangle| \leq 1\}.$$

**Lemma 0.3.5.** *Let  $(E, F, \langle -, - \rangle)$  be a duality and  $S \subseteq E$ . Then  $S^\circ$  is a convex subset of  $F$  containing 0 that is  $\sigma(F, E)$ -closed, and  $S^{|\circ|}$  is absolutely convex and  $\sigma(F, E)$ -closed.*

*Proof.* For the fact that  $S^\circ$  is convex, contains 0, and is  $\sigma(F, E)$ -closed, see [110, IV.1.4].

To show  $S^{|\circ|}$  is absolutely convex, let  $\sum_{i \in I} \alpha_i y_i$  be a finite absolutely convex combination of elements of  $S^{|\circ|}$ . Then for all  $x \in S$ , we have

$$\left| \left\langle x, \sum_{i \in I} \alpha_i y_i \right\rangle \right| = \left| \sum_{i \in I} \alpha_i \langle x, y_i \rangle \right| \leq \sum_{i \in I} |\alpha_i| |\langle x, y_i \rangle| \leq \sum_{i \in I} |\alpha_i| \leq 1.$$

This shows  $\sum_{i \in I} \alpha_i y_i \in S^{|\circ|}$ . Now, since  $S^{|\circ|} = S^\circ \cap -S^\circ$ , it is  $\sigma(F, E)$ -closed as well.  $\square$

**Lemma 0.3.6.** *Let  $(E, F, \langle -, - \rangle)$  be a duality. If  $S \subseteq E$  is absolutely convex, then  $S^\circ = S^{|\circ|}$ .*

*Proof.* We see from the definition that  $S^{|\circ|} \subseteq S^\circ$ . To show the opposite inclusion, let  $y \in S^\circ$ . We know that for all  $x \in S$ ,  $\langle x, y \rangle \leq 1$ . Since  $-x \in S$ , by absolute convexity, we have  $\langle -x, y \rangle \leq 1$ , so  $\langle x, y \rangle \geq -1$ , by bilinearity. This shows that  $|\langle x, y \rangle| \leq 1$ , for all  $x \in S$ , and hence  $y \in S^{|\circ|}$ .  $\square$

Recall that the polar wedge of a wedge  $C \subseteq E$  is

$$C^* = \{y \in F \mid \langle x, y \rangle \geq 0\}$$

**Lemma 0.3.7.** *Let  $(E, F, \langle -, - \rangle)$  be a duality, and  $C \subseteq E$  a wedge. Then  $C^* = -C^\circ$ .*

*Proof.* We have that

$$\begin{aligned} -C^\circ &= -\{y \in F \mid \forall x \in C. \langle x, y \rangle \leq 1\} = \{y \in F \mid \forall x \in C. \langle x, -y \rangle \leq 1\} \\ &= \{y \in F \mid \forall x \in C. \langle x, y \rangle \geq -1\}. \end{aligned}$$

We can see, therefore, that  $-C^\circ \subseteq C^*$ . Suppose for a contradiction that there is a  $y \in -C^\circ \setminus C^*$ . Then there is some  $x \in C$  such that  $-1 \leq \langle x, y \rangle < 0$ . Take

$\alpha = \langle x, y \rangle$ . Therefore  $-\frac{2}{\alpha} > 0$ , so  $-\frac{2}{\alpha}x \in C$  because it is a wedge. We can therefore see that

$$\left\langle -\frac{2}{\alpha}x, y \right\rangle = -\frac{2}{\alpha}\langle x, y \rangle = -2 \not\geq -1,$$

which is a contradiction. So  $-C^\circ \setminus C^* = \emptyset$ , and therefore  $-C^\circ = C^*$ .  $\square$

**Lemma 0.3.8.** *If  $E = E_+ - E_+$  (equivalently, if  $(E, E_+)$  is directed), then the dual wedge  $F_+$  is a cone, the dual cone.*

*Proof.* Suppose  $y \in F_+$  and  $-y \in F_+$ . Then for all  $x \in E_+$ , we have  $\langle x, y \rangle \geq 0$  and  $\langle x, -y \rangle \geq 0$ . By linearity, we deduce  $\langle x, y \rangle \leq 0$  and therefore  $\langle x, y \rangle = 0$  for all  $x \in E_+$ . If  $x \in E$  is expressed as  $x_+ - x_-$  with  $x_+, x_- \in E_+$ , we see that

$$\langle x, y \rangle = \langle x_+ - x_-, y \rangle = \langle x_+, y \rangle - \langle x_-, y \rangle = 0,$$

and therefore  $y = 0$ .  $\square$

The main theorem in the subject of polars is the following. Given a set  $S \subseteq E$ , we may not only take the polar  $S^\circ$ , but also the polar of the polar,  $S^{\circ\circ} \subseteq E$ , the *bipolar*.

**Theorem 0.3.9** (Bipolar Theorem). *Let  $(E, F, \langle -, - \rangle)$  be a duality,  $S \subseteq E$  a subset. Then  $S^{\circ\circ}$  is the closed convex hull of  $S \cup \{0\}$  in the  $\sigma(E, F)$  topology.*

*Proof.* See [110, IV.1.5] or [15, II.6.3 Theorem 1].  $\square$

**Corollary 0.3.10.** *Let  $(E, F, \langle -, - \rangle)$  be a duality, and  $S \subseteq E$  an absolutely convex set. Then  $S^{\circ\circ}$  is the  $\sigma(E, F)$  closure of  $S$ , or equivalently the closed absolutely convex hull of  $S$ .*

*Proof.* By absolute convexity, we have  $S \cup \{0\} = S$  and  $\text{co}(S \cup \{0\}) = S$ . By [20, IV.1.13 Corollary], we have that the closed convex hull of  $S$  is the closure of the convex hull of  $S$ , so by the bipolar theorem  $S^{\circ\circ} = \text{cl}(S)$ , in the  $\sigma(E, F)$  topology.

The closure of an absolutely convex set is convex, and if  $x_i \rightarrow x$ , then  $-x_i \rightarrow -x$ , so  $\text{cl}(S)$  is balanced and convex, therefore absolutely convex (Lemma A.3.1). Therefore  $\text{cl}(S)$  is the closed absolutely convex hull of  $S$ .  $\square$

**Lemma 0.3.11.** *Let  $(E, F, \langle -, - \rangle)$  be a duality.*

- (i) *Let  $(S_i)_{i \in I}$  be a family of subsets of  $E$ . Then  $\bigcap_{i \in I} S_i^\circ = \left(\bigcup_{i \in I} S_i\right)^\circ$ , and  $\bigcap_{i \in I} S_i^{|\circ|} = \left(\bigcup_{i \in I} S_i\right)^{|\circ|}$ .*



(ii) Let  $S \subseteq E$  and  $\alpha \in \mathbb{R} \setminus \{0\}$ . Then  $\alpha(S)^\circ = (\alpha^{-1}S)^\circ$  and  $\alpha(S)^{|\circ|} = (\alpha^{-1}S)^{|\circ|}$ .

(iii) Let  $S, T \subseteq E$ . Then  $S \subseteq T$  implies  $T^\circ \subseteq S^\circ$  and  $T^{|\circ|} \subseteq S^{|\circ|}$ .

(iv) Let  $S \subseteq E$ . Then  $\text{absco}(S)^\circ = \text{absco}(S)^{|\circ|} = S^{|\circ|}$ .

*Proof.* In each of the first three statements, we only give the argument for the polar, as the argument for the absolute polar is similar.

(i)

$$\begin{aligned} y \in \bigcap_{i \in I} S_i^\circ &\Leftrightarrow \forall i \in I. \forall x \in S_i. \langle x, y \rangle \leq 1 \Leftrightarrow \forall x \in \bigcup_{i \in I} S_i. \langle x, y \rangle \leq 1 \\ &\Leftrightarrow y \in \left( \bigcup_{i \in I} S_i \right)^\circ. \end{aligned}$$

(ii)

$$\begin{aligned} y \in \alpha(S^\circ) &\Leftrightarrow \alpha^{-1}y \in S^\circ \Leftrightarrow \forall x \in S. \langle x, \alpha^{-1}y \rangle \leq 1 \Leftrightarrow \forall x \in S. \langle \alpha^{-1}x, y \rangle \leq 1 \\ &\Leftrightarrow \forall x \in \alpha^{-1}S. \langle x, y \rangle \leq 1 \Leftrightarrow y \in (\alpha^{-1}S)^\circ \end{aligned}$$

(iii) Let  $S \subseteq T \subseteq E$ . Then

$$y \in T^\circ \Leftrightarrow \forall x \in T. \langle x, y \rangle \leq 1 \Rightarrow \forall x \in S. \langle x, y \rangle \leq 1 \Leftrightarrow y \in S^\circ.$$

(iv) By Lemma 0.3.6,  $\text{absco}(S)^\circ = \text{absco}(S)^{|\circ|}$ . We have  $S \subseteq \text{absco}(S)$ , so by the previous part  $\text{absco}(S)^{|\circ|} \subseteq S^{|\circ|}$ . To show the opposite inclusion, suppose that  $y \in S^{|\circ|}$ , i.e. that for all  $x \in S$ ,  $|\langle x, y \rangle| \leq 1$ . Let  $\sum_{i \in I} \alpha_i x_i$  be a finite absolutely convex combination of elements of  $S$ . Then

$$\begin{aligned} \left| \left\langle \sum_{i \in I} \alpha_i x_i, y \right\rangle \right| &= \left| \sum_{i \in I} \alpha_i \langle x_i, y \rangle \right| \\ &\leq \sum_{i \in I} |\alpha_i| |\langle x_i, y \rangle| \\ &= \sum_{i \in I} |\alpha_i| |\langle x_i, y \rangle| \\ &\leq \sum_{i \in I} |\alpha_i| && x_i \in S, y \in S^{|\circ|} \\ &\leq 1 && \text{an absolutely convex combination} \end{aligned}$$

□

**Corollary 0.3.12.** *If  $(E, F, \langle -, - \rangle)$  is a duality, and  $S \subseteq E$ , then  $S^{|\cdot| |\cdot|}$  is the closed absolutely convex hull of  $S$  in the  $\sigma(E, F)$  topology.*

*Proof.* By Lemma 0.3.11, we have  $S^{|\cdot| |\cdot|} = \text{absco}(S)^{|\cdot| |\cdot|}$ , which in turn is equal to  $\text{absco}(S)^{\circ\circ}$  by Lemma 0.3.6. By Corollary 0.3.10, this is the closed absolutely convex hull of  $S$  in the  $\sigma(E, F)$  topology. □

**Lemma 0.3.13.** *Let  $(E, F, \langle -, - \rangle)$  be a duality, and  $F' \subseteq F$  a subspace of  $F$  such that  $F$  separates the points of  $E$ , and therefore  $(E, F', \langle -, - \rangle)$  is a duality. Let  $S \subseteq E$ . We use  $S_F^\circ$  to mean the polar of  $S$  in  $F$  and  $S_{F'}^\circ$  to mean the polar of  $S$  in  $F'$ . Then*

$$S_F^\circ \cap F' = S_{F'}^\circ$$

*Proof.* If  $y \in S_{F'}^\circ$ , then  $y \in F'$  and  $\forall x \in S. \langle x, y \rangle \leq 1$ , so  $y \in S_F^\circ$ . Therefore  $y \in S_F^\circ \cap F'$ .

For the other direction, let  $y \in S_F^\circ \cap F'$ . Then  $y \in F'$  and  $\forall x \in S. \langle x, y \rangle \leq 1$ , so  $y \in S_{F'}^\circ$ . □

The following lemma relates polars and adjoints.

**Lemma 0.3.14.** *Let  $(E_1, F_1, \langle -, - \rangle_1)$  and  $(E_2, F_2, \langle -, - \rangle_2)$  be dualities,  $f : E_1 \rightarrow E_2$  a linear map with adjoint  $g : F_2 \rightarrow F_1$ . If  $S \subseteq E_1$ , then  $f(S)^\circ = g^{-1}(S^\circ)$ . Equivalently, if  $T \subseteq F_2$ , then  $g(T)^\circ = f^{-1}(T^\circ)$ .*

*Proof.* The second statement follows from the first by transposing the duality, so we prove the first.

$$\begin{aligned} f(S)^\circ &= \{y \in F_2 \mid \forall x \in f(S). \langle x, y \rangle_2 \leq 1\} \\ &= \{y \in F_2 \mid \forall x \in S. \langle f(x), y \rangle_2 \leq 1\} \\ &= \{y \in F_2 \mid \forall x \in S. \langle x, g(y) \rangle_1 \leq 1\} \\ &= \{y \in F_2 \mid g(y) \in S^\circ\} \\ &= g^{-1}(S^\circ). \end{aligned}$$

□

We can prove the following useful fact about closed wedges in locally convex spaces. It can be proven directly from the Hahn-Banach separation theorem as well.

**Lemma 0.3.15.** *Let  $E$  be a locally convex space and  $E_+ \subseteq E$  a closed wedge. Then  $\phi(x) \geq 0$  for all  $\phi \in E_+^*$  (the polar wedge), iff  $x \in E_+$ .*

*Proof.* If  $x \in E_+$ , by definition we have  $\phi(x) \geq 0$  for all  $\phi \in E_+^*$ . We therefore only need to show the other direction.

The set  $\{x \in E \mid \forall \phi \in E_+^*. \phi(x) \geq 0\}$  is equal to the dual cone of  $E_+^*$  with respect to the transpose of the usual pairing between  $E$  and  $E^*$ . Applying Lemma 0.3.7 twice, we have

$$\begin{aligned} \{x \in E \mid \forall \phi \in E_+^*. \phi(x) \geq 0\} &= -(-E_+^\circ)^\circ \\ &= E_+^{\circ\circ} \end{aligned} \quad \text{Lemma 0.3.11 (ii)}$$

The bipolar is the  $\sigma(E, E^*)$ -closure of  $E_+$  by the bipolar theorem and the fact that a wedge is already convex and contains 0. We then use the fact that if a convex set is closed in a locally convex space it is also weakly closed by Proposition 0.3.4.  $\square$

## 0.4 Category Theory

We recall here some basic theorems of category theory. Some basic references are [81, 14]. We use Eilenberg's notation for hom sets in a category, *i.e.* if  $\mathcal{C}$  is a (locally small) category, and  $X$  and  $Y$  are objects in  $\mathcal{C}$ , then  $\mathcal{C}(X, Y)$  is the set of arrows  $X \rightarrow Y$  in  $\mathcal{C}$ .

The basic definition of an adjunction is a pair of functors  $F : \mathcal{D} \rightarrow \mathcal{C}, G : \mathcal{C} \rightarrow \mathcal{D}$  and a natural isomorphism  $\phi : \mathcal{D}(F(X), Y) \Rightarrow \mathcal{C}(X, G(Y))$ . An adjunction can be defined equivalently in the following ways:

**Theorem 0.4.1.** *Each adjunction  $(F, G, \phi)$ ,  $F : \mathcal{D} \rightarrow \mathcal{C}$  is determined by any one of the following:*

- (i) *Functors  $F, G$ , a natural transformation  $\eta : \text{Id} \Rightarrow GF$  such that each  $\eta_X$  is a universal arrow from  $X$  to  $G$ , *i.e.* for each  $f : X \rightarrow GY$  there exists a unique  $g : FX \rightarrow Y$  such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GFX \\ & \searrow f & \downarrow Gg \\ & & GY. \end{array}$$

*In this case,  $\phi(f)$  is defined to be  $G(f) \circ \eta_X$ .*

- (ii) *The functor  $G$  and for each  $X \in \mathcal{D}$  an object  $F_0X \in \mathcal{C}$  and a universal arrow  $\eta_X : X \rightarrow GF_0X$  from  $X$  to  $G$ . Then  $F$  is defined on objects as  $F_0X$  and on maps  $h : X \rightarrow X'$  to be the unique  $Fh$  such that  $GFh \circ \eta_X = \eta_{X'} \circ h$ .*

- (iii) Functors  $F, G$  and a natural transformation  $\epsilon : FG \Rightarrow \text{Id}$  such that  $\epsilon_Y$  is universal from  $F$  to  $Y$ .
- (iv) The functor  $F$  and for each  $Y \in \mathcal{C}$  an object  $G_0Y \in \mathcal{D}$  and an arrow  $\epsilon_Y : FG_0Y \rightarrow Y$  that is universal from  $F$  to  $Y$ .
- (v) Functors  $F, G$  and natural transformations  $\eta : \text{Id} \Rightarrow GF$ ,  $\epsilon : FG \Rightarrow \text{Id}$  such that the following diagrams commute

$$\begin{array}{ccc}
 FX & \xrightarrow{F\eta_X} & FGFX \\
 & \searrow \text{id}_{FX} & \downarrow \epsilon_{FX} \\
 & & FX
 \end{array}
 \qquad
 \begin{array}{ccc}
 GY & \xrightarrow{\eta_{GY}} & GFGY \\
 & \searrow \text{id}_{GX} & \downarrow G\epsilon_Y \\
 & & GY
 \end{array}$$

for each  $X \in \mathcal{D}$  and  $Y \in \mathcal{C}$ .

*Proof.* See [81, IV.1 Theorem 2], where the statement of the theorem comes from.  $\square$

The following fact about adjoints justifies referring to “the” left adjoint or “the” right adjoint of a functor, at least up to isomorphism.

**Proposition 0.4.2.** *If  $F, F' : \mathcal{D} \rightarrow \mathcal{C}$  are both left adjoints to a functor  $G : \mathcal{C} \rightarrow \mathcal{D}$ , then  $F \cong F'$ . Similarly, if  $G, G'$  are both right adjoints to a functor  $F$ , then  $G \cong G'$ .*

*Proof.* See [81, §IV.1 Corollary 1]. This isomorphism can be defined in terms of the units  $\eta : \text{Id} \Rightarrow GF$ ,  $\eta' : \text{Id} \Rightarrow GF'$  and counits  $\epsilon : FG \Rightarrow \text{Id}$ ,  $\epsilon' : F'G \Rightarrow \text{Id}$  as  $\epsilon F' \circ F\eta' : F \rightarrow F'$  and  $\epsilon' F \circ F'\eta : F' \rightarrow F$ . These can be seen to be mutually inverse by using naturality and the triangle identities from Theorem 0.4.1 (v).  $\square$

A functor  $F : \mathcal{D} \rightarrow \mathcal{C}$  is an *equivalence* if there exist  $G : \mathcal{D} \rightarrow \mathcal{C}$  and natural isomorphisms  $\alpha : FG \Rightarrow \text{Id}_{\mathcal{C}}$ ,  $\beta : \text{Id}_{\mathcal{D}} \Rightarrow GF$ . We call the triple  $(G, \alpha, \beta)$  an *inverse* for  $F$ , and when no confusion can result, we refer to just  $G$  as an inverse. Be warned that there is no uniqueness to  $\alpha$  and  $\beta$ . An adjunction  $(F, G, \eta, \epsilon)$ , described in terms of (v) of the above theorem, is called an *adjoint equivalence* if  $\eta$  and  $\epsilon$  are isomorphisms. Recall that a functor  $F$  is *essentially surjective on objects* if for each  $Y \in \mathcal{C}$ , there exists an  $X \in \mathcal{D}$  and an isomorphism  $F(X) \cong Y$ .

**Theorem 0.4.3.** *The following are equivalent for a functor  $F : \mathcal{D} \rightarrow \mathcal{C}$ :*

- (i)  $F$  is an equivalence of categories.
- (ii)  $F$  is part of an adjoint equivalence  $(F, G, \eta, \epsilon)$  (so  $(G, F, \epsilon^{-1}, \eta^{-1})$  is an adjoint equivalence too).
- (iii)  $F$  is full, faithful and essentially surjective on objects.

*Proof.* See [81, IV.4 Theorem 1]. □

For  $F$  an equivalence, we call any  $G$  in an adjoint equivalence an *adjoint inverse* to  $F$ .

**Lemma 0.4.4.** *Let  $(F, G, \eta, \epsilon)$  be an adjoint equivalence, and let  $(F', G, \eta', \epsilon')$  be an adjunction (having the same  $G$ ). Then  $\eta'$  and  $\epsilon'$  are isomorphisms, so  $(F', G, \eta', \epsilon')$  is also an adjoint equivalence (with  $F' \cong F$ ).*

*Proof.* Using Proposition 0.4.2, we have that  $\epsilon F' \circ F \eta' : F \Rightarrow F'$  and  $\epsilon' F \circ F' \eta : F' \Rightarrow F$  form an isomorphism  $F \cong F'$ . We first observe that

$$\begin{aligned}
 G(\epsilon F' \circ F \eta') \circ \eta &= G\epsilon F' \circ GF\eta' \circ \eta \\
 &= G\epsilon F' \circ \eta GF' \circ \eta' && \text{naturality of } \eta \\
 &= (G\epsilon \circ \eta G)F' \circ \eta' \\
 &= \eta' && \text{adjunction triangle.}
 \end{aligned}$$

As  $\eta$  is an isomorphism,  $\eta'$  has been shown to be a composite of two isomorphisms, and therefore an isomorphism.

Similarly the proof for  $\epsilon'$  goes

$$\begin{aligned}
 \epsilon \circ (\epsilon' F \circ F' \eta)G &= \epsilon \circ \epsilon' FG \circ F' \eta G \\
 &= \epsilon' \circ F' G \epsilon \circ F' \eta G \\
 &= \epsilon' \circ F'(G\epsilon \circ \eta G) \\
 &= \epsilon',
 \end{aligned}$$

using the naturality of  $\epsilon'$  and an adjunction triangle.

It follows, by definition, that  $(F', G, \eta', \epsilon')$  is an adjoint equivalence. □

### 0.4.1 Monads

This section recalls the basics of the theory of monads. Some basic references are [81, 10, 84, 14]. Some specific examples are elaborated later on.

A monad is a functor  $T : \mathcal{C} \rightarrow \mathcal{C}$  together with two natural transformations: a unit  $\eta : \text{Id}_{\mathcal{C}} \Rightarrow T$  and a multiplication  $\mu : T^2 \Rightarrow T$ , such that the following diagrams commute, for  $X \in \mathcal{C}$ .

$$\begin{array}{ccc}
 T(X) & \xrightarrow{\eta_{T(X)}} & T^2(X) \xleftarrow{T(\eta_X)} T(X) \\
 & \searrow & \downarrow \mu_X \\
 & & T(X)
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^3(X) & \xrightarrow{\mu_{T(X)}} & T^2(X) \\
 T(\mu_X) \downarrow & & \downarrow \mu_X \\
 T^2 & \xrightarrow{\mu_X} & T(X)
 \end{array}$$

Each adjunction  $F \dashv G$  gives rise to a monad  $(GF, \eta, G\epsilon F)$ .

Given a monad  $T$  one can form the category  $\mathcal{EM}(T)$  of (Eilenberg-Moore) algebras. Objects of this category are maps of the form  $\alpha : T(X) \rightarrow X$ , making the first two diagrams below commute.

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & TX \\
 & \searrow & \downarrow \alpha \\
 & & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^2X & \xrightarrow{T(\alpha)} & TX \\
 \mu_X \downarrow & & \downarrow \alpha \\
 TX & \xrightarrow{\alpha} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 TX & \xrightarrow{T(f)} & TY \\
 \alpha \downarrow & & \downarrow \beta \\
 X & \xrightarrow{f} & Y
 \end{array}$$

A homomorphism of algebras  $(X, \alpha) \rightarrow (Y, \beta)$  is a map  $f : X \rightarrow Y$  in  $\mathcal{C}$  between the underlying objects making the diagram on the above right commute. Therefore the diagram in the middle says that the map  $\alpha$  is a homomorphism  $(TX, \mu_X) \rightarrow (X, \alpha)$ . The forgetful functor  $U : \mathcal{EM}(T) \rightarrow \mathcal{C}$  has a left adjoint, mapping an object  $X \in \mathcal{X}$  to the (free) algebra  $(T(X), \mu_X)$ .

Each category  $\mathcal{EM}(T)$  inherits limits from the category  $\mathcal{C}$ . In the special case where  $\mathcal{C} = \mathbf{Set}$ , the category of sets and functions, the category  $\mathcal{EM}(T)$  is not only complete but also cocomplete (see [10, § 9.3, Prop. 4]).

For any monad  $T = (T, \eta, \mu)$  on a category  $\mathbf{B}$  we write  $\mathcal{Kl}(T)$  for the Kleisli category of  $T$ . Its objects are the same as those of  $\mathbf{B}$ , but its maps  $X \rightarrow Y$  are the maps  $X \rightarrow T(Y)$  in  $\mathbf{B}$ . The unit  $\eta : X \rightarrow T(X)$  is the identity map  $X \rightarrow X$  in  $\mathcal{Kl}(T)$ ; and composition of  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in  $\mathcal{Kl}(T)$  is given by  $g \circ f = \mu \circ T(g) \circ f$ . Maps in such a Kleisli category are understood as computations with outcomes of type  $T$ , see [87]. For a monad  $T : \mathbf{Set} \rightarrow \mathbf{Set}$  we write  $\mathcal{Kl}_{\mathbb{N}}(T) \hookrightarrow \mathcal{Kl}(T)$  for the full subcategory with numbers  $n \in \mathbb{N}$  as objects, considered as  $n$ -element sets.

For a given monad  $(T, \eta, \mu)$ , it is natural to ask if it arises from an adjunction. The Kleisli and Eilenberg-Moore categories both show that this is true.

For  $\mathcal{Kl}(T)$  we define  $F_T : \mathcal{C} \rightarrow \mathcal{Kl}(T)$  and  $G_T : \mathcal{Kl}(T) \rightarrow \mathcal{C}$  as follows

$$\begin{aligned} F_T(X) &= X \\ F_T(f : X \rightarrow Y) &= \eta_Y \circ f \\ G_T(X) &= T(X) \\ G_T(f : X \rightarrow T(Y)) &= \mu_Y \circ T(f). \end{aligned}$$

Then  $F_T$  is a left adjoint to  $G_T$  in such a way that  $T$  is equal to the monad arising from this adjunction [81, §VI.5 Theorem 1].

For  $\mathcal{EM}(T)$ , we define  $F^T : \mathcal{C} \rightarrow \mathcal{EM}(T)$  and  $G^T : \mathcal{EM}(T) \rightarrow \mathcal{C}$  as

$$\begin{aligned} F^T(X) &= (TX, \mu_X) \\ F^T(f : X \rightarrow Y) &= T(f) \\ G^T(X, \alpha : TX \rightarrow X) &= X \\ G^T(f : X \rightarrow Y) &= f. \end{aligned}$$

Then  $F^T$  is a left adjoint to  $G^T$ , and  $T$  is the monad arising from this adjunction [81, §VI.2 Theorem 1].

If a monad  $(T : \mathcal{C} \rightarrow \mathcal{C}, \eta, \mu)$  is known to arise from an adjunction  $(F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}, \eta, \epsilon)$ , one might wonder how the previous adjunctions are related to the one defining  $T$ . There is a *comparison functor*  $K_T : \mathcal{Kl}(T) \rightarrow \mathcal{D}$  defined as

$$\begin{aligned} K_T(X) &= F(X) \\ K_T(f : X \rightarrow T(Y)) &= \epsilon_{F(Y)} \circ F(f). \end{aligned}$$

This functor is a map of adjunctions, *i.e.*  $G \circ K_T = G_T$  and  $K_T \circ F_T = F$ , and it is the unique functor with this property [81, §VI.5 Theorem 2].

There is also a comparison functor for  $\mathcal{EM}(T)$   $K^T : \mathcal{D} \rightarrow \mathcal{EM}(T)$  defined as

$$\begin{aligned} K^T(X) &= (G(X), G(\epsilon_X)) \\ K^T(f : X \rightarrow Y) &= G(f). \end{aligned}$$

This functor is also a map of adjunctions, *i.e.*  $G^T \circ K^T = G$  and  $K^T \circ F = F^T$  and is the unique functor with this property [81, §VI.3 Theorem 1].

The above constructions give us  $K^T, K_T : \mathcal{Kl}(T) \rightarrow \mathcal{EM}(T)$  for any monad  $T$ . These two functors are equal by the uniqueness statements above, and can

be defined as

$$\begin{aligned} K : \mathcal{Kl}(T) &\rightarrow \mathcal{EM}(T) \\ K(X) &= T(X) \\ K(f : X \rightarrow T(Y)) &= \mu_Y \circ T(f). \end{aligned}$$

If  $\mathcal{C}$  and  $\mathcal{D}$  are categories,  $(T, \eta^T, \mu^T)$ ,  $(S, \eta^S, \mu^S)$  are monads on  $\mathcal{C}$  and  $\mathcal{D}$  respectively, a *lax map of monads* [79, §6.1]<sup>3</sup> from  $T$  to  $S$  is a pair  $(U, \sigma)$  where  $U$  is a functor  $\mathcal{C} \rightarrow \mathcal{D}$  and  $\sigma$  is a natural transformation  $SU \rightarrow UT$  (note the reversal of direction) such that the following diagrams commute

$$\begin{array}{ccc} U \xrightarrow{\eta^S U} SU & & S^2 U \xrightarrow{S\sigma} SUT \xrightarrow{\sigma T} UT^2 \\ \searrow U\eta^T \quad \downarrow \sigma & & \downarrow \mu^S U \quad \downarrow U\mu^T \\ & UT & SU \xrightarrow{\sigma} UT. \end{array} \quad (0.4)$$

In the case that the  $U = \text{Id}$ , a lax map of monads  $T \rightarrow S$  is a *monad morphism*  $S \rightarrow T$ , see e.g. [14, Volume 2, Definition 4.5.8].

The following is also proven in [79, Lemma 6.1.1].

**Proposition 0.4.5.** *For each lax map of monads  $(U : \mathcal{C} \rightarrow \mathcal{D}, \sigma : SU \Rightarrow UT)$  we can define a functor  $U^\sigma : \mathcal{EM}(T) \rightarrow \mathcal{EM}(S)$  as*

$$\begin{aligned} U^\sigma(X, \alpha) &= (UX, U\alpha \circ \sigma_X) \\ U^\sigma(f : (X, \alpha) \rightarrow (Y, \beta)) &= U(f), \end{aligned}$$

where  $(X, \alpha), (Y, \beta)$  are  $T$ -algebras,  $f$  a  $T$ -algebra map between them.

*Proof.* We first show that  $U^\sigma(X, \alpha)$  is an  $S$ -algebra. We do this by pasting diagrams as follows:

$$\begin{array}{ccc} UX \xrightarrow{\eta_{UX}^S} SUX & & S^2 UX \xrightarrow{S\sigma_X} SUTX \xrightarrow{SU\alpha} SUX \\ \searrow U\eta_X^T \quad \downarrow \sigma_X & & \downarrow \sigma_{TX} \quad \downarrow \sigma_X \\ & UTX & UT^2 X \xrightarrow{UT\alpha} UTX \\ \searrow U\text{id}_X \quad \downarrow U\alpha & & \downarrow U\mu_X^T \quad \downarrow U\alpha \\ & UX & SUX \xrightarrow{\sigma_X} UTX \xrightarrow{U\alpha} UX \end{array}$$

<sup>3</sup>Originally called a *monad functor* in [115, §1].



On the left, the top triangle is from (0.4), the bottom one is  $U$  applied to the triangle diagram for  $X$  being a  $T$ -algebra. On the right, the left pentagonal part is from (0.4), the upper right square is naturality of  $\sigma$  and the bottom right square is  $U$  applied to the square diagram for  $X$  being a  $T$ -algebra.

We also need to show that if  $f$  is a  $T$ -algebra map, then  $U^\sigma(f)$  is an  $S$ -algebra map. We do this by pasting diagrams again:

$$\begin{array}{ccc}
 SUX & \xrightarrow{SUf} & SUY \\
 \sigma_X \downarrow & & \downarrow \sigma_Y \\
 UTX & \xrightarrow{UTf} & UTY \\
 U\alpha \downarrow & & \downarrow U\beta \\
 UX & \xrightarrow{Uf} & UY.
 \end{array}$$

The top square commutes by naturality of  $\sigma$ , and the bottom square is  $U$  applied to the square that commutes because  $f$  is a  $T$ -algebra map.

Preservation of identities and composition for  $U^\sigma$  follows directly from the fact that  $U$  is a functor.  $\square$

A *colax map of monads*  $T \rightarrow S$  is a pair  $(U : \mathcal{C} \rightarrow \mathcal{D}, \sigma : UT \Rightarrow SU)$  (note the direction is not reversed this time) such that

$$\begin{array}{ccc}
 U \xrightarrow{U\eta^T} UT & & UT^2 \xrightarrow{\sigma T} SUT \xrightarrow{S\sigma} S^2U \\
 \searrow \eta^S U & \downarrow \sigma & \downarrow U\mu^T \\
 & SU & UT \xrightarrow{\sigma} SU.
 \end{array} \quad (0.5)$$

This definition comes from [79, §6.1]. We see that the diagrams are simply those from (0.4) with  $\sigma$  reversed. The following also comes from [79, §6.1].

**Proposition 0.4.6.** *For each colax map of monads  $(U : \mathcal{C} \rightarrow \mathcal{D}, \sigma : UT \Rightarrow SU)$  we can define a functor  $U_\sigma : \mathcal{Kl}(T) \rightarrow \mathcal{Kl}(S)$  as*

$$\begin{aligned}
 U_\sigma(X) &= U(X) \\
 U_\sigma(f : X \rightarrow T(Y)) &= \sigma_Y \circ U(f).
 \end{aligned}$$

*Proof.* First observe that for a map  $f : X \rightarrow TY$ , we have that  $U_\sigma(f) : U(X) \rightarrow SU(Y)$ , and is therefore a  $\mathcal{Kl}(S)$ -map from  $U_\sigma(X)$  to  $U_\sigma(Y)$ , as is required.

To show that  $U_\sigma$  preserves identity maps, observe

$$U_\sigma(\eta_X^T) = \sigma_X \circ U\eta_X^T = \eta_{UX}^S,$$

by the left diagram in (0.5).

For composition, let  $f : X \rightarrow TY$  and  $g : Y \rightarrow TZ$ . If we write  $*$  for Kleisli composition, we have

$$\begin{aligned} U_\sigma(g * f) &= U_\sigma(\mu_Z^T \circ T(g) \circ f) \\ &= \sigma_Z \circ U\mu_Z^T \circ UT(g) \circ U(f) \\ &= \mu_{UZ}^S \circ S\sigma_Z \circ \sigma_{TZ} \circ UT(g) \circ U(f) && \text{right diagram (0.5)} \\ &= \mu_{UZ}^S \circ S\sigma_Z \circ SU(g) \circ \sigma_Y \circ U(f) && \text{naturality of } \sigma \\ &= \mu_{UZ}^S \circ SU_\sigma(g) \circ U_\sigma(f) \\ &= U_\sigma(g) * U_\sigma(f). \end{aligned}$$

□

A *weak map of monads*  $T \rightarrow S$  is a lax map of monads  $(U, \sigma : SU \Rightarrow UT)$  such that  $\sigma$  is an isomorphism. Therefore  $(U, \sigma^{-1})$  is a colax map of monads  $T \rightarrow S$  as well. We therefore have functors  $U^\sigma : \mathcal{EM}(T) \rightarrow \mathcal{EM}(S)$  and  $U_{\sigma^{-1}} : \mathcal{KL}(T) \rightarrow \mathcal{KL}(S)$ .

**Proposition 0.4.7.** *For a weak map of monads  $(U, \sigma : SU \Rightarrow UT)$ , the natural isomorphism  $\sigma$  makes the diagram*

$$\begin{array}{ccc} \mathcal{KL}(T) & \xrightarrow{K^T} & \mathcal{EM}(T) \\ U_{\sigma^{-1}} \downarrow & & \downarrow U^\sigma \\ \mathcal{KL}(S) & \xrightarrow{K^S} & \mathcal{EM}(S) \end{array}$$

*2-commute, i.e. it is an isomorphism  $\sigma : K^S U_{\sigma^{-1}} \Rightarrow U^\sigma K^T$ .*

*Proof.* On objects, where  $X \in \mathcal{C}$ , or equivalently  $X \in \mathcal{KL}(T)$ , we have that  $K^S(U_{\sigma^{-1}}(X)) = S(U(X))$  and  $U^\sigma(K^T(X)) = U(T(X))$ , so, as a family of maps,  $\tau_X : K^S U_{\sigma^{-1}}(X) \rightarrow U^\sigma K^T(X)$ . If we show that  $\tau$ , with this type, is natural, the fact that each  $\tau_X$  is an isomorphism will show that  $\tau$  is a natural isomorphism. Therefore we only need to show that for all  $f : X \rightarrow T(Y)$ , the

diagram

$$\begin{array}{ccc}
 K^S U_{\sigma^{-1}} X & \xrightarrow{\sigma_X} & U^\sigma K^T X \\
 \downarrow K^S U_{\sigma^{-1}}(f) & & \downarrow U^\sigma K^T(f) \\
 K^S U_{\sigma^{-1}} Y & \xrightarrow{\sigma_Y} & U^\sigma K^T Y
 \end{array}$$

commutes. Expanding the definitions, we see that

$$U^\sigma(K^T(f)) = U^\sigma(\mu_Y^T \circ T(f)) = U\mu_Y^T \circ UT(f)$$

and

$$K^S(U_{\sigma^{-1}}(f)) = K^S(\sigma_Y^{-1} \circ U(f)) = \mu_{UY}^S \circ S(\sigma_Y^{-1}) \circ SU(f).$$

After making these substitutions, the naturality diagram takes the form

$$\begin{array}{ccc}
 SUX & \xrightarrow{\sigma_X} & UTX \\
 \downarrow SU(f) & & \downarrow UT(f) \\
 SUTY & \xrightarrow{\sigma_{TY}} & UT^2Y \\
 \downarrow S(\sigma_Y^{-1}) & & \downarrow U(\mu_Y^T) \\
 S^2UY & & UTY \\
 \downarrow \mu_{UY}^S & \nearrow \sigma_Y & \\
 SUY & & 
 \end{array}$$

The top square commutes by naturality of  $\sigma$  in its original form, and the bottom pentagon commutes because  $\sigma$  is a lax map of monads (0.4), and using the invertibility of  $\sigma$ .  $\square$

In the special case of monad morphisms, we can form the category of monads on a given category  $\mathcal{C}$ ,  $\mathbf{Monad}(\mathcal{C})$ . Given a monad morphism  $\sigma : S \Rightarrow T$  (remembering the reversal of direction compared to lax maps of monads), we define  $\mathcal{EM}(\sigma) = \text{Id}^\sigma : \mathcal{EM}(T) \rightarrow \mathcal{EM}(S)$ . If we take  $\mathbf{Cat}$  to be the (superlarge) category with large categories as objects and functors as morphisms<sup>4</sup> we can prove the following fact.

**Proposition 0.4.8.** *With the above definition on morphisms,  $\mathcal{EM}$  defines a functor  $\mathbf{Monad}(\mathcal{C})^{\text{op}} \rightarrow \mathbf{Cat}$ .*

<sup>4</sup>Not any 2-category and not the large category of small categories.

*Proof.* We have already seen that  $\mathcal{EM}(T)$  is a category for  $T$  any monad and that  $\mathcal{EM}(f)$  is a functor for any monad morphism (Proposition 0.4.5). Therefore we only need to show that  $\mathcal{EM}$  preserves identities and composition.

On objects,  $\mathcal{EM}(\text{id}_T)(X, \alpha) = (X, \alpha \circ \text{id}_{TX}) = (X, \alpha)$ , and on maps,  $\mathcal{EM}(\sigma)$  for any monad morphism  $\sigma$  is equal to the identity functor by the definition of monad morphism as a special lax map of monads, so  $\mathcal{EM}(\text{id}_T) = \text{Id}_{\mathcal{EM}(T)}$ .

For composition, let  $\sigma : R \Rightarrow S$  and  $\tau : S \rightarrow T$  be monad morphisms. On objects, we have that

$$\begin{aligned} \mathcal{EM}(\tau)(\mathcal{EM}(\sigma)(X, \alpha)) &= \mathcal{EM}(\tau)(X, \alpha \circ \sigma) \\ &= (X, \alpha \circ \sigma \circ \tau) \\ &= \mathcal{EM}(\sigma \circ \tau)(X, \alpha). \end{aligned}$$

On maps we have that all three functors are the identity, hence agree. Therefore  $\mathcal{EM}(\sigma \circ \tau) = \mathcal{EM}(\tau) \circ \mathcal{EM}(\sigma)$  as functors, completing the proof that  $\mathcal{EM}$  is itself a functor  $\mathbf{Monad}(\mathcal{C}) \rightarrow \mathbf{Cat}$ .  $\square$

The previous proposition shows that isomorphic monads have isomorphic Eilenberg-Moore categories.

We have already seen that adjoints, if they exist, are unique up to natural isomorphism (Proposition 0.4.2). Here we need a stronger result, namely that there is also a monad isomorphism between the induced monads.

**Lemma 0.4.9.** *Consider a functor  $G : \mathbf{C} \rightarrow \mathbf{D}$  with two left adjoints:  $F \dashv G$  and  $F' \dashv G$ . The induced isomorphism  $F \cong F'$  also yields an isomorphism  $GF \cong GF'$  of monads on  $\mathbf{D}$ .*

*Proof.* Let's write  $\eta, \varepsilon$  for the unit and counit of the adjunction  $F \dashv G$ , and similarly  $\eta', \varepsilon'$  for  $F' \dashv G$ . The multiplication maps for the induced monads  $GF$  and  $GF'$  are then given by  $\mu_X = G(\varepsilon_{FX}) : GF GF(X) \rightarrow GF(X)$  and  $\mu'_X = G(\varepsilon'_{F'X})$ . There is then a natural isomorphism  $\sigma : F \Rightarrow F'$  with components:

$$\sigma_X = \left( F(X) \xrightarrow{F(\eta'_X)} FGF'(X) \xrightarrow{\varepsilon_{F'X}} F'(X) \right)$$

Then  $G\sigma : GF \Rightarrow GF'$  is a isomorphism of monads. By using the triangle identities we get:

$$\begin{aligned}
G\sigma \circ \eta &= G(\varepsilon F') \circ FG(\eta') \circ \eta \\
&= G(\varepsilon F') \circ \eta GF' \circ \eta' \\
&= \eta' \\
\mu' \circ G\sigma GF' \circ GFG\sigma &= G\varepsilon' F \circ G\varepsilon F' GF' \circ GF\eta' GF' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GFG\varepsilon' F' \circ GF\eta' GF' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GF(G\varepsilon' \circ \eta' G)F' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GF\eta' \circ G\varepsilon F \\
&= G\sigma \circ \mu.
\end{aligned}$$

□

### The Distribution Monad

We shall write  $\mathcal{D}$  for the discrete probability distribution monad on **Set**. It maps a set  $X$  to the set of formal convex combinations  $r_1x_1 + \cdots + r_nx_n$ , where  $x_i \in X$  and  $r_i \in [0, 1]$  with  $\sum_i r_i = 1$ . Alternatively,

$$\mathcal{D}(X) = \left\{ \varphi : X \rightarrow [0, 1] \mid \text{supp}(\varphi) \text{ is finite, and } \sum_{x \in X} \varphi(x) = 1 \right\},$$

where  $\text{supp}(\varphi) \subseteq X$  is the support of  $\varphi$ , containing all  $x$  with  $\varphi(x) \neq 0$ .

We also write  $\mathcal{D}_\infty$  for the infinite distribution monad defined as

$$\mathcal{D}_\infty(X) = \left\{ \phi : X \rightarrow [0, 1] \mid \sum_{x \in X} \phi(x) = 1 \right\}.$$

It follows from Lemma 0.1.9 that  $\text{supp}(\phi)$  is always countable. The functors  $\mathcal{D}, \mathcal{D}_\infty : \mathbf{Set} \rightarrow \mathbf{Set}$  form monads with the Dirac  $\delta$  function as the unit:

$$X \xrightarrow{\eta_X} \mathcal{D}X \qquad \mathcal{D}\mathcal{D}X \xrightarrow{\mu_X} \mathcal{D}X$$

$$x \longmapsto \delta_x = y \mapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \qquad \Psi \longmapsto \left( y \mapsto \sum_{\varphi \in \mathcal{D}X} \Psi(\varphi) \cdot \varphi(y) \right).$$

This monad is well-known and often occurs in the literature without attribution. Objects of the category  $\mathcal{EM}(\mathcal{D})$  of (Eilenberg-Moore) algebras of

this monad  $\mathcal{D}$  can be considered to be *abstract convex sets*, in interpreting the map  $\alpha : \mathcal{D}(X) \rightarrow X$  as taking a convex combination  $\sum_i r_i x_i$ . Morphisms then correspond to affine functions, preserving such convex sums, see [52]. In this thesis we also need to refer to convex subsets of vector spaces, so we have avoided using the term “convex set” for an object of  $\mathcal{EM}(\mathcal{D})$ . The earliest relation of  $\mathcal{D}$  to convex sets we could find in the literature is [117, 4.1.1], where  $\mathcal{D}$  is called  $G$ , Eilenberg-Moore algebras are called semiconvex sets, and the maps are called semiaffine maps.

The prime example of an Eilenberg-Moore algebra of  $\mathcal{D}$  is the unit interval  $[0, 1] \subseteq \mathbb{R}$  of probabilities. Also, for an arbitrary set  $X$ , the set of functions  $[0, 1]^X$ , or fuzzy predicates on  $X$ , is a convex set, via pointwise convex sums.

### The Ultrafilter Monad

A particular monad that plays an important role later in the thesis is the ultrafilter monad  $\mathcal{U} : \mathbf{Set} \rightarrow \mathbf{Set}$ , given by:

$$\begin{aligned} \mathcal{U}(X) &= \{ \mathcal{F} \subseteq \mathcal{P}(X) \mid \mathcal{F} \text{ is an ultrafilter} \} \\ &\cong \{ f : \mathcal{P}(X) \rightarrow \{0, 1\} \mid f \text{ is a homomorphism of Boolean algebras} \} \end{aligned}$$

where  $\{0, 1\}$  is the 2-element Boolean algebra. Such an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(X)$  satisfies, by definition, the following three properties.

- It is an up-set:  $V \supseteq U \in \mathcal{F} \Rightarrow V \in \mathcal{F}$ ;
- It is closed under finite intersections:  $X \in \mathcal{F}$  and  $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$ ;
- For each set  $U$  either  $U \in \mathcal{F}$  or  $\neg U = \{x \in X \mid x \notin U\} \in \mathcal{F}$ , but not both. As a consequence,  $\emptyset \notin \mathcal{F}$ .

For a function  $f : X \rightarrow Y$  one obtains  $\mathcal{U}(f) : \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$  by:

$$\mathcal{U}(f)(\mathcal{F}) = \{ V \subseteq Y \mid f^{-1}(V) \in \mathcal{F} \}.$$

Taking ultrafilters is a monad, with unit  $\eta : X \rightarrow \mathcal{U}(X)$  given by principal ultrafilters:

$$\eta(x) = \{ U \subseteq X \mid x \in U \}.$$

The multiplication  $\mu : \mathcal{U}^2(X) \rightarrow \mathcal{U}(X)$  is:

$$\mu(\mathcal{A}) = \{ U \subseteq X \mid D(U) \in \mathcal{A} \} \quad \text{where } D(U) = \{ \mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F} \}.$$

The set  $\mathcal{U}(X)$  of ultrafilters on a set  $X$  is a topological space with basic (compact) clopens given by subsets  $D(U) = \{ \mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F} \}$ , for  $U \subseteq X$ .

This makes  $\mathcal{U}(X)$  into a compact Hausdorff space. The unit  $\eta: X \rightarrow \mathcal{U}(X)$  is a dense embedding.

In fact,  $\mathcal{U}(X)$ , with this compact Hausdorff topology, defines a left adjoint to the forgetful functor  $\mathbf{CHaus} \rightarrow \mathbf{Set}$ , where  $\mathbf{CHaus}$  is the category of compact Hausdorff spaces and continuous maps, the full subcategory of  $\mathbf{Top}$ , the category of topological spaces and continuous maps.

The following result, that this adjunction is monadic, shows the importance of the ultrafilter monad, see *e.g.* [83], [81, VI.9], [62, III.2], or [14, Vol. 2, Prop. 4.6.6].

**Theorem 0.4.10** (Manes).  $\mathcal{EM}(\mathcal{U}) \simeq \mathbf{CHaus}$ , *i.e. the category of algebras of the ultrafilter monad is equivalent to the category  $\mathbf{CHaus}$  of compact Hausdorff spaces and continuous maps.*

The proof is complicated and will not be reproduced here. We only extract the basic constructions. For a compact Hausdorff space  $Y$  one uses denseness of the unit  $\eta$  to define a unique continuous extensions  $f^\#$  as in:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathcal{U}(X) \\ & \searrow f & \downarrow f^\# \\ & & Y \end{array}$$

One defines  $f^\#(\mathcal{F})$  to be the unique element in  $\bigcap \{\overline{V} \mid f^{-1}(V) \in \mathcal{F}\}$ . This intersection is a singleton precisely because  $Y$  is a compact Hausdorff space. In such a way one obtains an algebra  $\text{ch}_Y: \mathcal{U}(Y) \rightarrow Y$  as extension of the identity.

Conversely, given an algebra  $\text{ch}_X: \mathcal{U}(X) \rightarrow X$  one defines  $U \subseteq X$  to be closed if for all  $\mathcal{F} \in \mathcal{U}(X)$ ,  $U \in \mathcal{F}$  implies  $\text{ch}(\mathcal{F}) \in U$ . This yields a topology on  $X$  which is Hausdorff and compact. There can be at most one such algebra structure  $\text{ch}_X: \mathcal{U}(X) \rightarrow X$  on a set  $X$  corresponding to a compact Hausdorff topology, because of the following standard result.

**Lemma 0.4.11.** *Let  $X$  be a set with two topologies  $\mathcal{O}_1(X), \mathcal{O}_2(X) \subseteq \mathcal{P}(X)$  with  $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$ ,  $\mathcal{O}_1(X)$  is Hausdorff and  $\mathcal{O}_2(X)$  is compact, then  $\mathcal{O}_1(X) = \mathcal{O}_2(X)$ .*  $\square$

*Proof.* If  $U$  is closed in  $\mathcal{O}_2(X)$ , then it is compact, and, because  $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$ , also compact in  $\mathcal{O}_1(X)$ . Hence it is closed there.  $\square$

We can apply this result to the space  $\mathcal{U}(X)$  of ultrafilters: as described before Theorem 0.4.10,  $\mathcal{U}(X)$  carries a compact Hausdorff topology with base  $D(U) = \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}$  of clopens. Since it is a free  $\mathcal{U}$ -algebra by

the map  $\mu_X: \mathcal{U}^2(X) \rightarrow \mathcal{U}(X)$ , it has compact Hausdorff topology by Manes's theorem. It is not hard to see that the subsets  $D(U)$  are closed in the latter topology, so the two topologies on  $\mathcal{U}(X)$  coincide by Lemma 0.4.11.

**Example 0.4.12.** *The unit interval  $[0, 1] \subseteq \mathbb{R}$  is a standard example of a compact Hausdorff space. Its Eilenberg-Moore algebra  $\text{ch}: \mathcal{U}([0, 1]) \rightarrow [0, 1]$  can be described concretely on  $\mathcal{F} \in \mathcal{U}([0, 1])$  as:*

$$\text{ch}(\mathcal{F}) = \inf\{s \in [0, 1] \mid [0, s] \in \mathcal{F}\}. \quad (0.6)$$

*For the proof, recall that  $\text{ch}(\mathcal{F})$  is the sole element of  $\bigcap\{\overline{V} \mid V \in \mathcal{F}\}$ . Hence if  $[0, s] \in \mathcal{F}$ , then  $\text{ch}(\mathcal{F}) \in \overline{[0, s]} = [0, s]$ , so  $\text{ch}(\mathcal{F}) \leq s$ . This establishes the  $(\leq)$ -part of (0.6). Assume next that  $\text{ch}(\mathcal{F}) < \inf\{s \mid [0, s] \in \mathcal{F}\}$ . Then there is some  $r \in [0, 1]$  with  $\text{ch}(\mathcal{F}) < r < \inf\{s \mid [0, s] \in \mathcal{F}\}$ . Then  $[0, r]$  is not in  $\mathcal{F}$ , so that  $\neg[0, r] = (r, 1] \in \mathcal{F}$ . But this means  $\text{ch}(\mathcal{F}) \in \overline{(r, 1]} = [r, 1]$ , which is impossible.*

*Notice that (0.6) can be strengthened to*

$$\text{ch}(\mathcal{F}) = \inf\{s \in [0, 1] \cap \mathbb{Q} \mid [0, s] \in \mathcal{F}\}.$$



# Chapter 1

## C\*-algebras, Probability and Monads

*The following chapter, apart from Section 1.6, originated as “From Kleisli Categories to Commutative C\*-algebras: Probabilistic Gelfand Duality” [43] and its journal version [44]. The introduction to effect modules and the expectation monad comes from [60].*

### 1.1 Introduction

There are several notions of computation. We have the classical notion of computation, probabilistic computation, where a computer may make random choices, and quantum computation, which uses quantum mechanical interference and measurement. Normally we would consider classical computation to be done on sets, probabilistic computation on some kind of spaces admitting a notion of probability measures, and quantum computation on Hilbert spaces. We can instead use categories with C\*-algebras as objects and a choice of either \*-homomorphisms (called MIU-map below) or positive unital maps as the morphisms. The general outline is represented in this table.

	set-theoretic	probabilistic	quantum
$C^*$ -algebras	commutative	commutative	non-commutative
maps preserve	multiplication involution unit	positivity unit	positivity unit
maps abbreviation	MIU	PU	PU

We note at this point that positive unital maps coincide with *completely* positive unital maps if either the domain or codomain of a map is a commutative  $C^*$ -algebra, but not in general. While the quantum case is an important source of motivation, we will deal mainly with the classical and probabilistic cases in this chapter. In particular, we will relate the alternative method of representing probabilistic computation, using monads, to the  $C^*$ -algebraic approach.

In recent years the methods and tools of category theory have been applied to Hilbert spaces — see *e.g.* [1] and the references there — and also to  $C^*$ -algebras, see for instance [93, 86]. In this chapter we relate the distinction between different types of homomorphisms of  $C^*$ -algebras to the distinction between different types of computation. Moreover, we demonstrate the relevance of monads (and their Kleisli and Eilenberg-Moore categories) in this field. The aforementioned paper [93] concerns itself with only the  $*$ -homomorphisms (*a.k.a.* the MIU-maps).

The main results of this chapter can be summarized as follows. The well-known finite (‘baby’) version of Gelfand duality involves an equivalence between the category of finite sets (and all functions between them), and the opposite of the category of finite-dimensional commutative  $C^*$ -algebras, with MIU-maps ( $*$ -homomorphisms) between them. Diagrammatically:

$$\mathbf{FinSet} \xrightarrow{\simeq} \mathbf{FdCC}^*\mathbf{Alg}^{\mathrm{op}}$$

Our first observation is that if we generalize from MIU to PU (positive unital) maps we get an equivalence:

$$\mathcal{Kl}_{\mathbb{N}}(\mathcal{D}) \xrightarrow{\simeq} \mathbf{FdCC}^*\mathbf{Alg}_{\mathrm{PU}}^{\mathrm{op}}$$

where  $\mathcal{D}$  is the distribution monad on  $\mathbf{Set}$ , and  $\mathcal{Kl}_{\mathbb{N}}(\mathcal{D})$  is the Kleisli category of this monad, but with objects restricted to natural numbers. This shows that the category  $\mathbf{FdCC}^*\mathbf{Alg}_{\mathrm{PU}}$  is equivalent to the Lawvere theory of the distribution monad. The details are in Section 1.4.

The main contribution of this chapter lies in a generalization of the latter equivalence beyond the finite case<sup>1</sup>, which can be summarized in a diagram:

$$\begin{array}{ccc}
 \begin{array}{c} \mathcal{R} \\ \curvearrowright \\ \mathbf{CHaus} \end{array} & \xrightarrow[\text{Gelfand}]{\simeq} & \mathbf{CC^*Alg}^{\text{op}} \\
 \begin{array}{c} \uparrow \dashv \\ \mathcal{Kl}(\mathcal{R}) \end{array} & \xrightarrow[\text{new}]{\simeq} & \mathbf{CC^*Alg}_{\text{PU}}^{\text{op}}
 \end{array} \tag{1.1}$$

At the top of this diagram we have the classical Gelfand duality between the category  $\mathbf{CHaus}$  of compact Hausdorff spaces and the (opposite of the) category of commutative  $C^*$ -algebras with MIU-maps. Again, the generalization to the computationally more interesting PU-maps involves a duality with a Kleisli category, namely the Kleisli category  $\mathcal{Kl}(\mathcal{R})$  of what we call the Radon monad  $\mathcal{R}$  on compact Hausdorff spaces. By the Riesz representation theorem, elements of  $\mathcal{R}(X)$  can be described as Radon probability measures, which in this case coincide with inner regular probability measures (see [107, Theorem 2.14]).

The closest results in the literature to (1.1) are Umegaki's theorem in [119, Theorem 7.1] relating Baire-measurable Markov kernels on compact Hausdorff spaces to positive unital maps  $\mathcal{L}^\infty(Y) \rightarrow \mathcal{L}^\infty(X)$ , and Kozen's results in [75, §2] working on arbitrary measurable spaces. As these results, and much of the probabilistic literature, use Markov kernels rather than Kleisli maps, we discuss the relationship between these things and the aforementioned results in Section 1.6.

Incidentally, the adjunction on the left in Diagram (1.1) can be transferred to the right, and then yields a right adjoint to the inclusion  $\mathbf{CC^*Alg} \hookrightarrow \mathbf{CC^*Alg}_{\text{PU}}$ . In [122] it is shown that such a right adjoint also exists in the general non-commutative case.

Giry [46, I.4] described how we can consider a stochastic process as being a diagram in the Kleisli category of the Giry monad on measurable spaces. By using the Radon monad  $\mathcal{R}$  on compact spaces instead, we can get a different category of stochastic processes on compact spaces as diagrams in the (opposite of the) category of *commutative*  $C^*$ -algebras with PU-maps. This suggests the quantum generalization, considering diagrams in the category of non-commutative  $C^*$ -algebras. The use of the Kleisli category of  $\mathcal{R}$  also suggests that one could generalize to Eilenberg-Moore algebras of  $\mathcal{R}$ . We will see in chapters 3 and 4 how these two ideas are related.

<sup>1</sup>Though this can also be proved from a theorem of Umegaki, see Section 1.6.

We also show that the category of  $C^*$ -algebras with PU-maps embeds fully and faithfully in the category of effect modules, an algebraic structure for predicates adapted to quantum mechanics. At the end of Section 1.5, we then show, by considering monad morphisms, that Eilenberg-Moore algebras of  $\mathcal{R}$  and  $\mathcal{E}$  are compact Hausdorff spaces admitting an abstract convex structure.

## 1.2 Preliminaries on $C^*$ -algebras

We write  $\mathbf{Vect} = \mathbf{Vect}_{\mathbb{C}}$  for the category of vector spaces over the complex numbers  $\mathbb{C}$ . This category has direct product  $V \oplus W$ , forming a biproduct (both a product and a coproduct) and tensors  $V \otimes W$ , which distribute over  $\oplus$ . The tensor unit is the space  $\mathbb{C}$  of complex numbers. The unit for  $\oplus$  is the singleton (null) space 0. We write  $\bar{V}$  for the vector space with the same vectors/elements as  $V$ , but with conjugate scalar product:  $z \bullet_{\bar{V}} v = \bar{z} \bullet_V v$ . This makes  $\mathbf{Vect}$  an involutive category, see [53].

A  $*$ -algebra is an involutive monoid  $A$  in the category  $\mathbf{Vect}$ . Thus,  $A$  is itself a vector space, carries a multiplication  $\cdot : A \otimes A \rightarrow A$ , linear in each argument, and has a unit  $1 \in A$ . Moreover, there is an involution map  $(-)^* : \bar{A} \rightarrow A$ , preserving 0 and  $+$  and satisfying:

$$\begin{aligned} 1^* &= 1 & (x \cdot y)^* &= y^* \cdot x^* \\ x^{**} &= x & (z \bullet x)^* &= \bar{z} \bullet x^*. \end{aligned}$$

Here we have written a fat dot  $\bullet$  for scalar multiplication, to distinguish it from the algebra's multiplication  $\cdot$ . For  $z = a + bi \in \mathbb{C}$  we have the conjugate  $\bar{z} = a - bi$ . Often we omit the multiplication dot  $\cdot$  and simply write  $xy$  for  $x \cdot y$ . Similarly, the scalar multiplication  $\bullet$  is often omitted.

An element  $a$  of a  $*$ -algebra  $A$  is called *self-adjoint* if  $a^* = a$ . The  $\mathbb{R}$ -linearity of the  $-^*$  operation shows that self-adjoint elements form a real subspace of  $A$ . If  $a, b$  are self-adjoint, then  $ab$  is self-adjoint iff  $ab = ba$  because  $(ab)^* = b^*a^*$ .

A  $C^*$ -algebra is a  $*$ -algebra  $A$  with a norm  $\| - \| : A \rightarrow \mathbb{R}_{\geq 0}$  in which it is complete, satisfying the conditions  $\|x\| = 0$  iff  $x = 0$  and:

$$\begin{aligned} \|x + y\| &\leq \|x\| + \|y\| & \|z \bullet x\| &= |z| \cdot \|x\| \\ \|x \cdot y\| &\leq \|x\| \cdot \|y\| & \|x^* \cdot x\| &= \|x\|^2. \end{aligned}$$

The last equation  $\|x^* \cdot x\| = \|x\|^2$ , is the  $C^*$ -identity and distinguishes  $C^*$ -algebras from Banach  $*$ -algebras. We remark at this point that a Banach  $*$ -algebra admits at most one norm satisfying the  $C^*$ -identity. The reason for

this is that the spectral radius  $r(x)$  is definable in terms of the ring structure of the algebra, and for self-adjoint elements  $r(x) = \|x\|$  [63, Proposition 4.1.1 (a)]. If  $x$  is an arbitrary element,  $x^* \cdot x$  is self-adjoint, so  $r(x^* \cdot x) = \|x^* \cdot x\| = \|x\|^2$ . In the current setting, each  $C^*$ -algebra is unital, *i.e.* has a (multiplicative) unit 1. A consequence of the axioms above is that  $\|1\| = 1$  unless the  $C^*$ -algebra is the unique one in which  $0 = 1$ . A  $C^*$ -algebra is called *commutative* if its multiplication is commutative, and *finite-dimensional* if it has finite dimension as a vector space.

An element  $x$  in a  $C^*$ -algebra  $A$  is called *positive* if it can be expressed as  $x = y^* \cdot y$ . We write  $A^+ \subseteq A$  for the subset of positive elements in  $A$ . This subset is a cone, which is to say it is closed under addition and scalar multiplication with positive real numbers, and  $A_+ \cap -A_+ = \{0\}$  [26, Proposition 1.6.1]. Positive elements are self-adjoint, and we can deduce from this that the product of two positive elements is positive iff they commute. The square  $x^2 = x \cdot x$  of a self-adjoint element  $x = x^*$  is obviously positive. The positive cone defines an order on every  $C^*$ -algebra by (0.1), this is the usual order on a  $C^*$ -algebra.

We will consider three options when it comes to maps between  $C^*$ -algebras. The difference between them plays an important role in this chapter.

**Definition 1.2.1.** *We define three categories  $\mathbf{C^*Alg}$ ,  $\mathbf{C^*Alg}_{PU}$  and  $\mathbf{C^*Alg}_{P \leq 1}$  with  $C^*$ -algebras as objects, but with different morphisms. We also define their full subcategories  $\mathbf{CC^*Alg}$ ,  $\mathbf{CC^*Alg}_{PU}$  and  $\mathbf{CC^*Alg}_{P \leq 1}$  on commutative  $C^*$ -algebras.*

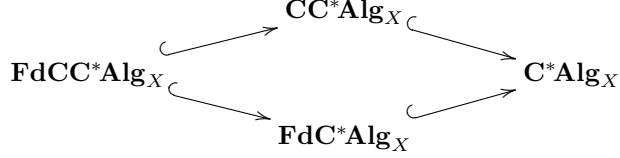
- (i) *A morphism  $f: A \rightarrow B$  in  $\mathbf{C^*Alg}$  is a linear map preserving multiplication ( $M$ ), involution ( $I$ ), and unit ( $U$ ). Explicitly, this means for all  $x, y \in A$ ,*

$$f(x \cdot y) = f(x) \cdot f(y) \qquad f(x^*) = f(x)^* \qquad f(1) = 1.$$

*Such “MIU” maps are usually called  $*$ -homomorphisms.*

- (ii) *A morphism  $f: A \rightarrow B$  in  $\mathbf{C^*Alg}_{PU}$  is a linear map that preserves positive elements and the unit. This means that  $f$  restricts to a function  $A^+ \rightarrow B^+$ . Alternatively, for each  $x \in A$  there is a  $y \in B$  with  $f(x^*x) = y^*y$ .*
- (iii) *A morphism  $f: A \rightarrow B$  in  $\mathbf{C^*Alg}_{P \leq 1}$  is a linear map that preserves positive elements and maps the unit  $1_A$  to some element  $\leq 1_B$ , necessarily positive.*  $\square$

For all  $X \in \{MIU, PU, P \leq 1\}$  there are the obvious full subcategories of commutative and/or finite-dimensional  $C^*$ -algebras, as described in:



Clearly, each “MIU” map is also a “PU” map, and every “PU” map is subunital, so that we have inclusions  $\mathbf{C}^*\mathbf{Alg} \hookrightarrow \mathbf{C}^*\mathbf{Alg}_{PU} \hookrightarrow \mathbf{C}^*\mathbf{Alg}_{P \leq 1}$ , and also for the various subcategories. A map that preserves positive elements is called positive itself; and a unit preserving map is called unital. Positive unital maps are the natural notion of morphism between order unit spaces and Riesz spaces.

The special case in which the codomain is  $\mathbb{C}$  is important. We define sets of *states* and *multiplicative states* as:

$$\text{Stat}(A) = \mathbf{C}^*\mathbf{Alg}_{PU}(A, \mathbb{C}) \quad \text{and} \quad \text{MStat}(A) = \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}).$$

There is also the commonly used notion of completely positive maps, which is a stronger condition than positivity but weaker than being MIU. These maps are important when defining the tensor product of  $C^*$ -algebras as a functor, as the extension of positive maps to the tensor product need not be positive. They are also widely considered to represent the physically realizable transformations. Positive, but non-completely positive maps of  $C^*$ -algebras also have their uses, as entanglement witnesses for example [51, theorem 2]. In general, throughout this thesis we put complete positivity to one side, hoping that it can be added later via a general construction, as is sketched in [45, §4]. In this chapter, we mainly consider the commutative case, where positive and completely positive coincide anyway. In fact, since a completely positive unital map is what is known as a channel in quantum information, then Theorem 1.5.1 shows that every channel in Mislove’s sense [85] is a channel in the usual sense.

We collect some basic (standard) properties of PU-morphisms between  $C^*$ -algebras.

**Lemma 1.2.2.**

- (i) An element  $a$  in a  $C^*$ -algebra  $A$  can be expressed as  $a_{\Re} + ia_{\Im}$  with  $a_{\Re}$  and  $a_{\Im}$  self-adjoint, defined by the formulas

$$a_{\Re} = \frac{a + a^*}{2} \quad \quad a_{\Im} = \frac{a - a^*}{2i}.$$

*This decomposition is unique, i.e. any decomposition into real and imaginary self-adjoint parts is the same, and we have  $\|a_{\Re}\|, \|a_{\Im}\| \leq \|a\|$ .*

- (ii) *Any self-adjoint  $a$  can be expressed as  $a_+ - a_-$  where  $a_+, a_- \in A_+$  and  $\|a_+\|, \|a_-\| \leq \|a\|$ . We can arrange that  $a_+a_- = 0$ .*

*Proof.*

- (i) We see that  $a_{\Re}$  and  $a_{\Im}$  are self-adjoint. We have

$$a_{\Re} + ia_{\Im} = \frac{a + a^*}{2} + i \frac{a - a^*}{2i} = \frac{a + a^* + a - a^*}{2} = \frac{2a}{2} = a.$$

We now show uniqueness. Suppose that  $a = b_{\Re} + b_{\Im}$  with  $b_{\Re}$  and  $b_{\Im}$  self-adjoint. Then

$$a_{\Re} = \frac{a + a^*}{2} = \frac{b_{\Re} + ib_{\Im} + b_{\Re} - ib_{\Im}}{2} = \frac{2b_{\Re}}{2} = b_{\Re}.$$

The proof that  $a_{\Im} = b_{\Im}$  is similar.

For the inequality, we see:

$$\|a_{\Re}\| = \left\| \frac{a + a^*}{2} \right\| \leq \frac{1}{2}(\|a\| + \|a^*\|) = \|a\|,$$

and the argument for  $\|a_{\Im}\|$  is similar.

- (ii) See any of the following references: [63, Proposition 4.2.3 (iii)] [109, Defn. 1.4.3] [26, §1.5.7 and 1.6.5].  $\square$

**Lemma 1.2.3.** *A PU-map, i.e. a morphism in  $\mathbf{C}^*\mathbf{Alg}_{\text{PU}}$ , preserves self-adjointness of elements, commutes with involution  $(-)^*$ , and preserves the partial order  $\leq$  given by (0.1) (page 20).*

*Moreover, a PU-map  $f$  satisfies  $\|f(x)\| \leq 4\|x\|$ , so that  $\|f(x) - f(y)\| \leq 4\|x - y\|$ , making  $f$  continuous. In fact, this constant can be reduced to 1, i.e.  $\|f(x)\| \leq \|x\|$ .*

*As every unital  $*$ -homomorphism is a PU-map, the above facts are also true of all unital  $*$ -homomorphisms.*

*Proof.* Let  $f : A \rightarrow B$  be a PU map. By Lemma 1.2.2, if  $a \in A$  is self-adjoint, we have  $a = a_+ - a_-$ , where  $a_+, a_-$  are positive, so  $f(a) = f(a_+) - f(a_-)$ , which is a difference of two positive elements, and therefore a self-adjoint element.

If  $a$  is a general element, it can be expressed as  $a_{\Re} + ia_{\Im}$ ,  $a_{\Re}, a_{\Im}$  being self adjoint. We therefore have

$$\begin{aligned} f(a^*) &= f((a_{\Re} + ia_{\Im})^*) = f(a_{\Re} - ia_{\Im}) = f(a_{\Re}) - if(a_{\Im}) \\ &= (f(a_{\Re}) + if(a_{\Im}))^* = f(a_{\Re} + ia_{\Im})^* = f(a)^*. \end{aligned}$$

Preservation of the partial order is implied by preservation of positive elements.

For positive  $a$  we have  $a \leq \|a\| \bullet 1$ , and thus  $f(a) \leq \|a\| \bullet 1$ , which gives  $\|f(a)\| \leq \|a\|$ . An arbitrary element  $a \in A$  can be written as linear combination of four positive elements  $x_i$ , as in  $x = x_1 - x_2 + ix_3 - ix_4$ , with  $\|x_i\| \leq \|x\|$ . Finally,  $\|f(x)\| = \|f(x_1) - f(x_2) + if(x_3) - if(x_4)\| \leq \sum_i \|f(x_i)\| \leq \sum_i \|x_i\| \leq 4\|x\|$ .

The reduction of the constant to 1 follows from the Russo-Dye theorem [108, Corollary 1].  $\square$

We next recall two well-known adjunctions involving compact Hausdorff spaces. The first one is due to Manes [83] and describes compact Hausdorff spaces as monadic over **Set**, via the ultrafilter monad (see Theorem 0.4.10). The second one is known as Gelfand duality, relating compact Hausdorff spaces and *commutative*  $C^*$ -algebras. Notice that this result involves the “MIU” maps, *i.e.*  $*$ -homomorphisms. In the following theorem, recall that  $A^*$  refers to the vector space of *continuous* linear functionals, as defined in the beginning of Section 0.3.

**Theorem 1.2.4.** *Let **CHaus** be the category of compact Hausdorff spaces, with continuous maps between them. There are two fundamental adjunctions:*

$$\begin{array}{ccc} \mathbf{CHaus} & & \mathbf{CHaus} \\ \mathcal{U} \left( \dashv \right) \text{forget} & & C \left( \simeq \right) \text{Spec} \\ \mathbf{Set} & & \mathbf{CC^*Alg}^{\text{op}} \end{array}$$

On the left the functor  $\mathcal{U}$  sends a set  $X$  to the ultrafilters on the powerset  $\mathcal{P}(X)$ . And on the right the equivalence of categories is given by sending a compact Hausdorff space  $X$  to the commutative  $C^*$ -algebra  $C(X) = \mathbf{Top}(X, \mathbb{C})$  of continuous functions  $X \rightarrow \mathbb{C}$ . The underlying set of  $\text{Spec}(A)$  is  $\text{MStat}(A)$ , and the topology is the weak- $*$  topology  $\sigma(A^*, A)$ , as states are elements of  $A^*$  by Lemma 1.2.3.



The unit and counit of Gelfand duality are

$$\begin{aligned}\eta_X : X &\rightarrow \text{Spec}(C(X)) \\ \eta_X(x)(a) &= a(x) \\ \epsilon_A : C(\text{Spec}(A)) &\leftarrow A \\ \epsilon_A(a)(\phi) &= \phi(a)\end{aligned}$$

□

The multiplicative states on a commutative  $C^*$ -algebra can equivalently be described as maximal ideals, or also as pure states (see below).

**Corollary 1.2.5.** *For each finite-dimensional commutative  $C^*$ -algebra  $A$  there is an  $n \in \mathbb{N}$  with  $A \cong \mathbb{C}^n$  in  $\mathbf{FdCC}^*\mathbf{Alg}$ .*

*Proof.* By the previous theorem there is a compact Hausdorff space  $X$  such that  $A$  is MIU-isomorphic to the algebra of continuous maps  $X \rightarrow \mathbb{C}$ . This  $X$  must be finite, and since a finite Hausdorff space is discrete, all maps  $X \rightarrow \mathbb{C}$  are continuous. Let  $n \in \mathbb{N}$  be the number of elements in  $X$ ; then we have an isomorphism  $A \cong \mathbb{C}^n$ . □

As we can already see in the above theorem, it is the *opposite* of a category of  $C^*$ -algebras that provides the most natural setting for computations. This is in line with what is often called the Heisenberg picture. In a logical setting it corresponds to computation of weakest preconditions, going backwards. The situation may be compared to the category of frames<sup>2</sup>, which is most usefully known in opposite form, as the category of locales, see [62].

The set of states  $\text{Stat}(A) = \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(A, \mathbb{C})$  can be equipped with the weak-\* topology, which is the coarsest (smallest) topology in which all evaluation maps  $\text{ev}_x = \phi \mapsto \phi(x) : \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(A, \mathbb{C}) \rightarrow \mathbb{C}$ , for  $x \in A$ , are continuous. We introduce the category  $\mathbf{CCL}$ , which first appeared in [116], in order to extend  $\text{Stat}$  to a functor.

The category  $\mathbf{CCL}$  has as its objects compact convex subsets of (Hausdorff) locally convex topological vector spaces. More accurately, the objects are pairs  $(V, X)$  where  $V$  is a (Hausdorff) locally convex topological vector space, and  $X$  is a compact convex subset of  $V$ . The maps  $(V, X) \rightarrow (W, Y)$  are continuous, affine maps  $X \rightarrow Y$ . Note that if  $(V, X)$  and  $(W, Y)$  are isomorphic, while  $X$  is necessarily homeomorphic to  $Y$ ,  $V$  need not bear any particular relation to  $W$  at all. We can see  $\mathbf{CCL}$  forms a category, as identity

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<sup>2</sup>Complete Heyting algebras, but where frame homomorphisms are maps preserving infinite joins and finite meets.

maps are affine and continuous and both of these attributes of a map are preserved under composition. We remark at this point that we have a forgetful functor  $U: \mathbf{CCL} \rightarrow \mathbf{CHaus}$ , taking the underlying compact Hausdorff space of  $X$ .

**Proposition 1.2.6.** *For a  $C^*$ -algebra  $A$ , the states  $\text{Stat}(A) = \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(A, \mathbb{C})$  form a convex, compact Hausdorff subspace of the dual space of  $A$  given the weak-\* topology. Each PU-map  $f: A \rightarrow B$  yields an affine continuous function  $\text{Stat}(f) = (-) \circ f: \text{Stat}(B) \rightarrow \text{Stat}(A)$ . This defines a functor*

$$\text{Stat}: \mathbf{C}^*\mathbf{Alg}_{\text{PU}}^{\text{op}} \rightarrow \mathbf{CCL}.$$

We recall that a function (between convex sets) is called *affine* if it preserves convex sums. As we saw in section 0.4.1 such affine maps are homomorphisms of Eilenberg-Moore algebras for the distribution monad  $\mathcal{D}$ .

*Proof.* For each finite collection  $h_i \in \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(A, \mathbb{C})$  with  $r_i \in [0, 1]$  satisfying  $\sum_i r_i = 1$ , the function  $h = \sum_i r_i h_i$  is again a state. Moreover, such convex sums are preserved by precomposition, making the maps  $(-) \circ f$  affine.

The fact that the dual space of  $A$ , given the weak-\* topology, is a locally convex space is standard (Proposition 0.3.1 and after). This implies that the space of states is Hausdorff. The set of positive linear functionals is defined to be the dual cone of the positive operators, so is closed (Lemmas 0.3.7 and 0.3.5) and the set of linear functionals such that  $\phi(1) = 1$  is weak-\* closed, and the set of states is the intersection of the two, and therefore closed. The space of states is also bounded as each state has norm 1. Therefore the state space is a closed and bounded and hence compact by the Banach-Alaoglu Theorem.

Precomposition  $(-) \circ f$  is continuous, since for  $x \in A$  and  $U \subseteq \mathbb{C}$  open we get an open subset  $((-) \circ f)^{-1}(\text{ev}_x^{-1}(U)) = \{h \mid \text{ev}_x(h \circ f) \in U\} = \text{ev}_{f(x)}^{-1}(U)$ .

Precomposition with the identity map gives the same state again, so  $\text{Stat}$  preserves identity maps. Since composition of PU-maps is associative,  $\text{Stat}$  preserves composition, and so is a functor.  $\square$

### 1.2.1 Effect modules

This section introduces effect modules and notions related to them, referring to [52, 58, 59]. Intuitively, effect modules are like vector spaces, but instead of  $\mathbb{R}$  as scalars, we have  $[0, 1]$ , and instead of an underlying abelian group, they have an underlying effect algebra. Effect modules were introduced as “convex effect algebras” in [48].

Effect algebras were introduced in mathematical physics, in the investigation of quantum probability, see [40, 29]. An *effect algebra* is a partial commutative monoid  $(M, 0, \odot)$  with an orthocomplement  $(-)^{\perp}$ . One writes  $x \perp y$  if  $x \odot y$  is defined. Commutativity of  $\odot$  needs to be defined in such a way that existence of  $a \odot b$  implies the existence of  $b \odot a$ , and an analogous condition is also necessary for associativity. The orthocomplement satisfies  $x^{\perp\perp} = x$  and  $x \odot x^{\perp} = 1$ , where  $1 = 0^{\perp}$ . We also require that  $a \odot b = 1$  implies  $b = a^{\perp}$ , uniqueness of orthocomplement. On any effect algebra there is always a partial order, given by  $x \leq y$  iff there exists a  $z$  such that  $x \odot z = y$ . Our main example of an effect algebra is the unit interval  $[0, 1] \subseteq \mathbb{R}$ , where addition  $+$  is made partial,  $a + b$  being defined only if the sum is in  $[0, 1]$ . This is commutative, associative, and has 0 as a unit; moreover, the orthocomplement is  $r^{\perp} = 1 - r$ . We write **EA** for the category of effect algebras, where the morphisms are maps preserving  $\odot$  and 1 — and thus all other structure.

For each set  $X$ , the set  $[0, 1]^X$  of fuzzy predicates on  $X$  is an effect algebra, via pointwise operations. Each Boolean algebra  $B$  is an effect algebra with  $x \perp y$  iff  $x \wedge y = \perp$ ; then  $x \odot y = x \vee y$ . In a quantum setting, the motivating example is the set of effects  $\text{Ef}(\mathcal{H}) = \{E: \mathcal{H} \rightarrow \mathcal{H} \mid 0 \leq E \leq I\}$  on a Hilbert space  $\mathcal{H}$ , see e.g. [29, 50].

The category **EA** carries a symmetric monoidal structure  $\otimes$  with the 2-element effect algebra  $\{0, 1\}$  as tensor unit (which is at the same time the initial object), see [58]. The usual multiplication of real numbers (probabilities in this case) yields a monoid structure on  $[0, 1]$  in the category **EA**. An *effect module* is then an effect algebra with an  $[0, 1]$ -action  $[0, 1] \otimes E \rightarrow E$ . Explicitly, it can be described as a scalar multiplication  $(r, x) \mapsto rx$  satisfying:

$$\begin{aligned} 1x &= x & (r+s)x &= rx \odot sx & \text{if } r+s \leq 1 \\ (rs)x &= r(sx) & r(x \odot y) &= rx \odot ry & \text{if } x \perp y. \end{aligned}$$

In particular, if  $r+s \leq 1$ , then a sum  $rx \odot sy$  always exists (Lemma A.4.1, see also [48]).

The algebras  $[0, 1]^X$  and  $\text{Ef}(\mathcal{H})$  are clearly effect modules. Other examples of effect modules, occurring in integration theory, are the sets  $[X \rightarrow_s [0, 1]]$  of *simple functions*  $X \rightarrow [0, 1]$ , having only finitely many output values.

A morphism  $E \rightarrow D$  in the category **EMod** of effect modules is a function  $f: E \rightarrow D$  between the underlying sets satisfying:

$$\begin{aligned} f(rx) &= rf(x) & f(1) &= 1 \\ f(x \odot y) &= f(x) \odot f(y) & \text{if } x \perp y. \end{aligned}$$

Each effect module can be “totalized” to produce a partially ordered vector

space with a special kind of unit. We first explain totalization of effect algebras, from [58, Proposition 3].

**Proposition 1.2.7.** *There is a coreflection*

$$\mathbf{EA} \begin{array}{c} \xrightarrow{\tau} \\ \perp \\ \xleftarrow{[0,u]_{(-)}} \end{array} \mathbf{BCM} \quad (1.2)$$

where **BCM** is the category of “barred commutative monoids”: its objects are pairs  $(M, u)$ , where  $M$  is a commutative monoid and  $u \in M$  is a distinguished element, called the unit, such that  $x + y = 0$  implies  $x = y = 0$  and  $x + y = x + z = u$  implies  $y = z$ . The morphisms in **BCM** are monoid homomorphisms that preserve the unit. As this is a coreflection every effect algebra  $E$  is isomorphic to  $[0, u]_{\mathcal{T}(E)}$ .  $\square$

The partialization functor  $[0, u]_{(-)}$  in (1.2) is defined by the ‘unit interval’:

$$[0, u]_M = \{x \in M \mid x \preceq u\},$$

where  $x \preceq y$  iff there exists a  $z$  such that  $x + z = y$ . The operation  $\otimes$  is defined by  $x \otimes y = x + y$  but this is only defined if  $x + y \preceq u$ , i.e.  $x + y \in [0, u]_M$ .

The totalization of the effect algebra  $\{0, 1\}$  is the natural numbers  $\mathbb{N}$  with 1 as the unit, and the totalization of  $[0, 1]$  is  $\mathbb{R}_{\geq 0}$ , again with 1 as the unit. An important difference between these examples is that in  $\mathbb{N}$ , if we pick different non-zero elements as order units, we obtain non-isomorphic objects in **BCM** and non-isomorphic unit intervals. However, for  $\mathbb{R}_{\geq 0}$ , the choices of order unit are all isomorphic.

We can now discuss the totalization of effect modules. In an ordered vector space,  $(A, A_+)$ ,  $u \in A_+$  is a *strong order unit* if for all  $x \in A$ , there is some  $\alpha \in \mathbb{R}_{\geq 0}$  such that  $-\alpha u \leq x \leq \alpha u$ . It is equivalent to require that  $A_+$  be generating and that for all  $x \in A_+$  there is a  $\lambda$  such that  $x \leq \lambda u$ , by Lemma A.5.1. A triple  $(A, A_+, u)$  where  $(A, A_+)$  is an ordered vector space and  $u$  a strong order unit is called a *partially ordered vector space with unit* in [59, before Theorem 3], and the category **poVectu** has these as objects and the maps are linear positive maps preserving the unit. We call these maps (*positive*) *unital maps*, as in the  $C^*$ -algebraic case.

We define the unit ball of  $(A, A_+, u) \in \mathbf{poVectu}$  as  $U = [-u, u] = \{x \in A \mid -u \leq x \leq u\}$ . This is absolutely convex and absorbing, so its Minkowski functional  $\|\cdot\|_U$  defines a seminorm on  $E$ . We say that  $(A, A_+, u)$  is *archimedean* if  $x \leq \frac{1}{n}u$  for all  $n \in \mathbb{N}_{>0}$  implies  $x \in -A_+$ . This implies

that  $\|\cdot\|_U$  is a norm, in which the positive cone is closed (Lemma A.5.3). Be warned that the condition one might expect, that  $x \in A_+$  and  $x \leq \frac{1}{n}u$  for all  $n \in \mathbb{N}_{>0}$  implies  $x = 0$ , is strictly weaker than archimedeaness, and is itself weaker than the condition that  $\|\cdot\|_U$  be a norm, which is known as being almost archimedean [61, 1.3.7].

A partially ordered vector space with unit  $(A, A_+, u)$  is called an *order-unit space* if it is archimedean. If it is complete in its norm, then it is a *Banach order-unit space*. In [59, p.154 and Proposition 11] the categories **OUS** and **BOUS** are defined, having order-unit spaces and Banach order-unit spaces as objects (respectively), and with maps being positive maps that preserve the unit, *i.e.* as full subcategories of **poVectu**. We also define **OUS** $_{\leq 1}$  and **BOUS** $_{\leq 1}$ , the category of order-unit spaces and *subunital maps*, which are maps  $f : (E, E_+, u) \rightarrow (F, F_+, v)$  such that  $f(u) \leq v$ , and its full subcategory on Banach order-unit spaces.

We note at this point that we allow  $(\{0\}, \{0\}, 0)$  as an order-unit space.

**Proposition 1.2.8.** *If  $f : (E, E_+, u) \rightarrow (F, F_+, v)$  is subunital or unital,  $\|f\| \leq 1$ . If  $F \neq 0$  and  $f$  is unital,  $\|f\| = 1$ .*

*Proof.* To show that  $\|f\| \leq 1$ , it is sufficient to show that if  $U$  is the closed unit ball of  $E$  and  $V$  is the closed unit ball of  $F$ ,  $f(U) \subseteq V$ . Since  $U = [-u, u]$  and  $V = [-v, v]$ , all we need to show is that if  $-u \leq x \leq u$ , then  $-v \leq f(x) \leq v$ . Whether the map is taken to be unital or subunital, we have  $f(u) \leq v$ . By the positivity and linearity of  $f$ , we have

$$-v \leq f(-u) \leq f(x) \leq f(u) \leq v.$$

Now assume that  $F \neq 0$ . By Lemma A.5.2,  $v \neq 0$  and  $\|v\| = 1$ . Since  $f(u) = v$ , we must also have  $u \neq 0$ , or we would have  $v = 0$  by linearity, and so  $\|u\| = 1$  as well. Since  $f(u) = v$ ,  $f$  maps an element of norm 1 to an element of norm 1, so has operator norm at least 1. Since  $\|f\| \leq 1$ , we have  $\|f\| = 1$ .  $\square$

A particular consequence of the above is that every map, unital or subunital, of order-unit spaces is continuous, and isomorphisms between order-unit spaces are isometries of the underlying Banach spaces.

There are also full subcategories of **EMod** on *archimedean effect modules* and *Banach effect modules*, **AEMod** and **BEMod** respectively. These are defined in [59, pp. 154-155]. An effect module is archimedean if  $x \leq y$  holds when  $\frac{1}{2}x \leq \frac{1}{2}y \otimes \frac{r}{2}$  for all  $r \in (0, 1]$ , and a metric can be defined [59, (10)] on each archimedean effect module, and Banach effect modules are those that are complete in this metric.

**Theorem 1.2.9.** *The unit interval functor  $[0, 1]_- : \mathbf{poVectu} \rightarrow \mathbf{EMod}$  is an equivalence of categories, with  $\mathcal{T} : \mathbf{EMod} \rightarrow \mathbf{poVectu}$ . Restricting these functors gives adjoint equivalences  $\mathbf{OUS} \simeq \mathbf{AEMod}$  and  $\mathbf{BOUS} \simeq \mathbf{BEMod}$ .*

*Proof.* See [59, Theorem 3] and [59, Propositions 9, 11].  $\square$

We review our examples of effect modules: both the effect modules  $[0, 1]$  and  $[0, 1]^X$  are archimedean, and also Banach effect modules. Norms and distances in  $[0, 1]$  are the usual ones, but limits in  $[0, 1]^X$  are defined via the supremum (or uniform) norm.

For any  $C^*$ -algebra  $A$ , we can define  $\mathrm{SA}(A)$  to be the set of self-adjoint elements. This is an  $\mathbb{R}$ -subspace of  $A$ , and it is closed because it is equal to  $(\mathrm{id}_A - *)^{-1}(\{0\})$ , the preimage of a closed set under a continuous map. The positive cone  $A_+ \subseteq \mathrm{SA}(A)$ , and in fact  $\mathrm{SA}(A) = A_+ - A_+$  [26, §1.5.7 and 1.6.5], so  $(\mathrm{SA}(A), A_+)$  is a directed ordered vector space. The reader can probably see where this is going:

**Proposition 1.2.10.** *For each  $C^*$ -algebra  $A$ ,  $(\mathrm{SA}(A), A_+, 1_A)$  is a Banach order-unit space. If, for any PU-map  $f : A \rightarrow B$  we define  $\mathrm{SA}(f) = f|_{\mathrm{SA}(A)}$ , then  $\mathrm{SA}$  is a functor  $\mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}} \rightarrow \mathbf{BOUS}$ , and similarly for subunital maps we get a functor  $\mathbf{C}^*\mathbf{Alg}_{\mathrm{P} \leq 1} \rightarrow \mathbf{BOUS}_{\leq 1}$ . These functors are full and faithful.*

*Proof.* For the proof that  $(\mathrm{SA}(A), A_+, 1_A)$  is a Banach order-unit space, see [31, Proposition 5.2] or [6, Theorem 1.95], although undoubtedly the definition of order-unit space was motivated by  $\mathrm{SA}(A)$  in the first place. As positive maps preserve self-adjoint elements (Lemma 1.2.3), the map  $\mathrm{SA}(f)$  is well-defined, and its linearity, positivity and preservation of unit follow directly. Preservation of identities and composition by  $\mathrm{SA}$  is trivial.

To show that  $\mathrm{SA}$  is faithful, let  $f, g : A \rightarrow B$  be maps in  $\mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}}$  or  $\mathbf{C}^*\mathbf{Alg}_{\mathrm{P} \leq 1}$  (the proof is the same in either case) and suppose  $\mathrm{SA}(f) = \mathrm{SA}(g)$ . Then for any  $a \in A$ , applying Lemma 1.2.2 to express  $a = a_{\Re} + ia_{\Im}$ , we observe

$$\begin{aligned} f(a) &= f(a_{\Re} + ia_{\Im}) = \mathrm{SA}(f)(a_{\Re}) + i\mathrm{SA}(f)(a_{\Im}) = \mathrm{SA}(g)(a_{\Re}) + i\mathrm{SA}(g)(a_{\Im}) \\ &= g(a), \end{aligned}$$

so  $f = g$ .

To show that  $\mathrm{SA}$  is full, let  $f : \mathrm{SA}(A) \rightarrow \mathrm{SA}(B)$  be a positive subunital map. Using Lemma 1.2.2 again, define

$$g(a) = f(a_{\Re}) + if(a_{\Im})$$

for all  $a \in A$ .

To prove the additive part of linearity, consider  $a + b$ . We have that  $a_{\Re} + b_{\Re}$  is self-adjoint, as is the sum of the imaginary parts, and so by the uniqueness of the decomposition (Lemma 1.2.2)  $(a + b)_{\Re} = a_{\Re} + b_{\Re}$  and  $(a + b)_{\Im} = a_{\Im} + b_{\Im}$ . So

$$\begin{aligned} g(a + b) &= f(a_{\Re} + b_{\Re}) + if(a_{\Im} + b_{\Im}) = f(a_{\Re}) + f(b_{\Re}) + if(a_{\Im}) + if(b_{\Im}) \\ &= g(a) + g(b). \end{aligned}$$

For the multiplicative part of linearity, we first show it for multiplication by a real. Let  $\alpha \in \mathbb{R}$ . Since  $\alpha a_{\Re}$  and  $\alpha a_{\Im}$  are self-adjoint, they are the real and imaginary parts of  $\alpha a$ , so

$$g(\alpha a) = f(\alpha a_{\Re}) + if(\alpha a_{\Im}) = \alpha f(a_{\Re}) + i\alpha f(a_{\Im}) = \alpha g(a).$$

Now we show that  $g$  preserves multiplication by  $i$ . We see that  $(ia)_{\Re} = -a_{\Im}$  and  $(ia)_{\Im} = a_{\Re}$ , so

$$\begin{aligned} g(ia) &= f((ia)_{\Re}) + if((ia)_{\Im}) = f(-a_{\Im}) + if(a_{\Re}) = if(a_{\Re}) + i(f(a_{\Im})) \\ &= i(f(a_{\Re}) + if(a_{\Im})) = ig(a). \end{aligned}$$

We can now prove  $\mathbb{C}$ -linearity. Take  $z = \alpha + i\beta$ . Then

$$g(za) = g(\alpha a + i\beta a) = g(\alpha a) + g(i\beta a) = \alpha g(a) + i\beta g(a) = zg(a).$$

We have that  $g(1_A) = f(1_A) \leq 1_B$  by subunitality of  $f$ . If  $f$  is unital, then  $f(1_A) = 1_B$ , so  $g$  is unital. We also have that if  $a$  is positive, then its imaginary part is 0, so  $g(a) = f(a)$ , which is positive since  $f$  is a positive map. Thus we have fullness in both cases.  $\square$

We remark at this point that  $\text{SA}(A)$  for  $A$  a commutative  $C^*$ -algebra can be distinguished from  $\text{SA}(B)$  for  $B$  a non-commutative  $C^*$ -algebra because  $\text{SA}(A)$  is a lattice (without top or bottom) if and only if  $A$  is commutative. In the  $C^*$ -algebra  $B(\mathcal{H})$ , by comparison, is as far as possible from being a lattice, as two elements  $a, b \in \text{SA}(B(\mathcal{H}))$  have a join or meet iff they are comparable [65].

For a  $C^*$ -algebra  $A$  we can write  $[0, 1]_A \subseteq A^+ \subseteq A$  for the subset of positive elements below the unit. We see immediately that  $[0, 1]_A = [0, 1]_{\text{SA}(A)}$ . The elements in  $[0, 1]_A$  are known as *effects* (or sometimes also as *predicates*). This extends the definition of effect we saw before for  $B(\mathcal{H})$ , and we shall see in a moment that, in fact, it unifies this example with the other example  $[0, 1]^X$ , which is  $[0, 1]_{\ell^\infty(X)}$ .

Each PU-map of  $C^*$ -algebras  $f : A \rightarrow B$  preserves  $\leq$  and thus restricts to  $[0, 1]_A \rightarrow [0, 1]_B$ . This restriction is a map of effect modules. Hence we get a “predicate” functor  $\mathbf{C}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathbf{EMod}$ . This map is equal to  $[0, 1]_{\text{SA}(f)}$ . Therefore we have

**Corollary 1.2.11.** *The functor  $[0, 1]_{(-)} : \mathbf{C}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathbf{BEMod}$  is full and faithful.*

*Proof.*  $[0, 1]_{-} : \mathbf{BOUS} \rightarrow \mathbf{BEMod}$  is an equivalence by Theorem 1.2.9, and therefore full and faithful, and  $\text{SA} : \mathbf{C}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathbf{BOUS}$  is full and faithful by Proposition 1.2.10. Therefore their composite  $[0, 1]_{-} : \mathbf{C}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathbf{BEMod}$  is full and faithful.  $\square$

### 1.3 Set-theoretic Computations in $C^*$ -algebras

For a set  $X$ , a function  $f : X \rightarrow \mathbb{C}$  is called *bounded* if  $|f(x)| \leq s$ , for some  $s \in \mathbb{R}_{\geq 0}$ . We write  $\ell^\infty(X)$  for the set of such bounded functions. Notice that if  $X$  is finite, any function  $X \rightarrow \mathbb{C}$  is bounded, so that  $\ell^\infty(X) = \mathbb{C}^X$ .

Each  $\ell^\infty(X)$  is a commutative  $C^*$ -algebra, with pointwise addition, multiplication and involution, and with the uniform/supremum norm:

$$\|f\|_\infty = \inf\{s \in \mathbb{R}_{\geq 0} \mid \forall x. |f(x)| \leq s\}.$$

In fact it is a typical example of a commutative  $W^*$ -algebra, but we will leave  $W^*$ -algebras to Section 3.6. This yields a functor  $\ell^\infty : \mathbf{Set} \rightarrow \mathbf{CC}^*\mathbf{Alg}^{\text{op}}$ , where for  $h : X \rightarrow Y$  we have  $\ell^\infty(h) = (-) \circ h : \ell^\infty(Y) \rightarrow \ell^\infty(X)$ ; it preserves the (pointwise) operations. We have the following result.

**Proposition 1.3.1.** *The functor  $\ell^\infty : \mathbf{Set} \rightarrow \mathbf{CC}^*\mathbf{Alg}^{\text{op}}$  is left adjoint to the multiplicative states functor  $\text{MStat} : \mathbf{CC}^*\mathbf{Alg}^{\text{op}} \rightarrow \mathbf{Set}$ . In combination with the adjunctions from Theorem 1.2.4 we get the situation:*

$$\begin{array}{ccc}
 \mathbf{CHaus} & \begin{array}{c} \xrightarrow{C} \\ \xleftarrow{\sim} \\ \xrightarrow{\text{Spec}} \end{array} & \mathbf{CC}^*\mathbf{Alg}^{\text{op}} \\
 & \begin{array}{c} \nwarrow \perp \\ \nearrow \ell^\infty \end{array} & \\
 & \mathbf{Set} &
 \end{array}
 \quad
 \begin{array}{c}
 \swarrow \mathcal{U} \\
 \searrow \text{MStat}
 \end{array}$$

By composition and uniqueness of adjoints we get:

$$C \circ \mathcal{U} \cong \ell^\infty \qquad \text{and also} \qquad \text{Spec} \circ \ell^\infty \cong \mathcal{U}.$$



*Proof.* Recall that  $\mathbf{MStat}$  is the set underlying the compact Hausdorff space  $\text{Spec}$ . We first show  $\ell^\infty \dashv \mathbf{MStat}$  using by defining the unit and verifying the universal property (Theorem 0.4.1 (i)). We define the unit  $\eta_X: X \rightarrow \mathbf{MStat}(\ell^\infty(X))$ , where  $X \in \mathbf{Set}$ , as

$$\eta_X(x)(a) = a(x),$$

where  $a \in \ell^\infty(X)$ . Then  $\eta_X(x)$  is a multiplicative state on  $\ell^\infty(X)$  because the vector space structure, multiplication and multiplicative unit are defined pointwise. To show the naturality square for  $\eta$  commutes, we must show that for all  $f: X \rightarrow Y$  in  $\mathbf{Set}$ ,  $\mathbf{MStat}(\ell^\infty(f)) \circ \eta_X = \eta_Y \circ f$ . If we take  $x \in X$  and  $b \in \ell^\infty(Y)$ , we have:

$$\begin{aligned} (\mathbf{MStat}(\ell^\infty(f)) \circ \eta_X)(x)(b) &= \mathbf{MStat}(\ell^\infty(f))(\eta_X(x))(b) \\ &= (\eta_X(x) \circ \ell^\infty(f))(b) \\ &= \eta_X(x)(\ell^\infty(f)(b)) \\ &= \eta_X(x)(b \circ f) \\ &= b(f(x)) \\ &= \eta_Y(f(x))(b) \\ &= (\eta_Y \circ f)(x)(b). \end{aligned}$$

We now show this natural transformation satisfies the universal property making it the unit of the adjunction. Let  $X \in \mathbf{Set}$ ,  $B \in \mathbf{CC}^*\mathbf{Alg}$  and  $f: X \rightarrow \mathbf{MStat}(B)$ . Define  $g: B \rightarrow \ell^\infty(X)$  as  $g(b)(x) = f(x)(b)$ . We must show that  $g(b)$  is an element of  $\ell^\infty(X)$ , *i.e.* that it is bounded. For all  $x \in X$ ,  $f(x)$  is a multiplicative state, hence a state, so by [26, Proposition 2.1.4] we have  $\|f(x)\| = 1$ , and so  $|g(b)(x)| = |f(x)(b)| \leq \|f(x)\| \|b\| = \|b\|$ . Therefore  $\|b\|$  is a bound for  $g(b)$ , showing that it is a bounded function. The fact that  $g$  is an MIU map is easily deduced from the fact that  $f(x)$  is a multiplicative state for all  $x$  (it would fail if  $f(x)$  were only a state).

We must now show that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathbf{MStat}(\ell^\infty(X)) \\ & \searrow f & \downarrow \mathbf{MStat}(g) \\ & & \mathbf{MStat}(B) \end{array}$$

commutes. Taking  $x \in X$  and  $b \in B$ , we see

$$\begin{aligned} \mathbf{MStat}(g)(\eta_X(x))(b) &= (\eta_X(x) \circ g)(b) = \eta_X(x)(g(b)) \\ &= g(b)(x) = f(x)(b), \end{aligned}$$

and hence the unit diagram commutes.

To show the uniqueness of  $g$ , suppose there were  $h: B \rightarrow \ell^\infty(X)$  that also made the unit diagram commute. By evaluating  $\text{MStat}(h)(\eta_X(x))(b)$  we would obtain  $g(b)(x) = h(b)(x)$ . Since  $g(b)$  and  $h(b)$  are elements of  $\ell^\infty(X)$  and hence functions, this implies  $g(b) = h(b)$  by extensionality, and we can then conclude that  $g = h$ , as required. We have now shown that  $\ell^\infty$  is a left adjoint to  $\text{MStat}$ . The other two adjunctions are simply the Stone-Ćech compactification of a set and Gelfand duality (which is even an equivalence).

Since the triangle consisting of the forgetful functor  $\mathbf{CHaus} \rightarrow \mathbf{Set}$ ,  $\text{MStat}$  and  $\text{Spec}$  commutes, the triangle for  $\ell^\infty$ ,  $\mathcal{U}$  and  $C$  commutes up to isomorphism, i.e.  $\ell^\infty \cong C \circ \mathcal{U}$  by uniqueness of adjoints (Proposition 0.4.2).  $\square$

When we restrict to the full subcategory  $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$  of finite sets we obtain a functor  $\ell^\infty = \mathbb{C}^{(-)}: \mathbf{FinSet} \rightarrow \mathbf{FdCC}^*\mathbf{Alg}^{\text{op}}$ . The next result is then a well-known special case of Gelfand duality (Theorem 1.2.4). We elaborate the proof in some detail because it is important to see where the preservation of multiplication plays a role.

**Proposition 1.3.2.** *The functor  $\mathbb{C}^{(-)}: \mathbf{FinSet} \rightarrow \mathbf{FdCC}^*\mathbf{Alg}^{\text{op}}$  is an equivalence of categories.*

*Proof.* It is easy to see that the functor  $\mathbb{C}^{(-)}$  is faithful. The crucial part is to see that it is full. So assume we have two finite sets, seen as natural numbers  $n, m$ , and a MIU-homomorphism  $h: \mathbb{C}^m \rightarrow \mathbb{C}^n$ . For  $j \in m$ , let  $|j\rangle \in \mathbb{C}^m$  be the standard base vector with 1 at the  $j$ -th position and 0 elsewhere. Since this  $|j\rangle$  is positive, so is  $h(|j\rangle)$ , and thus we may write it as  $h(|j\rangle) = (r_{1j}, \dots, r_{nj})$ , with  $r_{ij} \in \mathbb{R}_{\geq 0}$ . Because  $|j\rangle \cdot |j\rangle = |j\rangle$ , and  $h$  preserves multiplication, we get  $h(|j\rangle) \cdot h(|j\rangle) = h(|j\rangle)$ , and thus  $r_{ij}^2 = r_{ij}$ . This means  $r_{ij} \in \{0, 1\}$ , so that  $h$  is a (binary) Boolean matrix. But  $h$  is also unital, and so:

$$1 = h(1) = h(|1\rangle + \dots + |m\rangle) = h(|1\rangle) + \dots + h(|m\rangle). \quad (1.3)$$

For each  $i \in n$  there is thus precisely one  $j \in m$  with  $r_{ij} = 1$  — so that  $h$  is a “functional” Boolean matrix. This yields the required function  $f: n \rightarrow m$  with  $\mathbb{C}^f = h$ .

Corollary 1.2.5 says that the functor  $\mathbb{C}^{(-)}: \mathbf{FinSet} \rightarrow \mathbf{FdCC}^*\mathbf{Alg}^{\text{op}}$  is essentially surjective on objects, and thus an equivalence.  $\square$

This proof demonstrates that preservation of multiplication, as required for “MIU” maps, is a rather strong condition. We make this more explicit.

**Corollary 1.3.3.** *For  $n \in \mathbb{N}$  we have  $\text{MStat}(\mathbb{C}^n) \cong n$ .*

*Proof.* By identifying  $n \in \mathbb{N}$  with the  $n$ -element set  $n = \{0, 1, \dots, n-1\}$ , we get by Proposition 1.3.2,

$$\mathbf{MStat}(\mathbb{C}^n) = \mathbf{C}^*\mathbf{Alg}(\mathbb{C}^n, \mathbb{C}) \cong \mathbf{FinSet}(1, n) \cong n.$$

□

## 1.4 Discrete Probabilistic Computations

We turn to probabilistic computations and will see that we remain in the world of commutative  $\mathbf{C}^*$ -algebras, but with PU-maps (positive unital) instead of MIU-maps. Recall that the set of states  $\mathbf{Stat}(A)$  of a  $\mathbf{C}^*$ -algebra  $A$  contains the PU-maps  $A \rightarrow \mathbb{C}$ .

We summarize here the definition of the expectation monad given in [59]. If  $[0, 1]^X$  is the effect module of functions from  $X$  to  $[0, 1]$  with pointwise operations,  $\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ . On maps, this is defined as

$$\mathcal{E}(f : X \rightarrow Y)(\phi \in \mathcal{E}(X))(b \in [0, 1]^Y) = \phi(b \circ f).$$

The unit  $\eta_X : X \rightarrow \mathcal{E}(X)$  is evaluation, defined as  $\eta_X(x)(a) = a(x)$  for a function  $a \in [0, 1]^X$ . The multiplication  $\mu_X : \mathcal{E}^2(X) \rightarrow \mathcal{E}(X)$  is defined for  $\Phi : [0, 1]^{\mathcal{E}(X)} \rightarrow [0, 1]$ ,  $a \in [0, 1]^X$  as

$$\mu_X(\Phi)(a) = \Phi(\phi \in \mathcal{E}(X) \mapsto \phi(a)). \quad (1.4)$$

This is proven to define a monad in [59, §4].

**Lemma 1.4.1.** *Sending a set  $X$  to the set of states of the  $\mathbf{C}^*$ -algebra  $\ell^\infty(X)$  yields the (underlying functor of the) expectation monad  $\mathcal{E}$  from [59]: the mapping  $X \mapsto \mathbf{Stat}(\ell^\infty(X))$  is isomorphic to the expectation monad  $\mathcal{E} : \mathbf{Set} \rightarrow \mathbf{Set}$ , defined in [59] via effect module morphisms:  $\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ .*

*As a result,  $\mathbf{Stat}(\mathbb{C}^n) \cong \mathcal{D}(n)$ , for  $n \in \mathbb{N}$ , where  $\mathcal{D}(n)$  is the standard  $(n-1)$ -simplex.*

*Proof.* The predicate/effect functor  $[0, 1]_{(-)} : \mathbf{C}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathbf{EMod}$  is full and faithful by Lemma 1.2.11, and so:

$$\begin{aligned} \mathbf{Stat}(\ell^\infty(X)) &= \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(\ell^\infty(X), \mathbb{C}) \cong \mathbf{EMod}([0, 1]_{\ell^\infty(X)}, [0, 1]_{\mathbb{C}}) \\ &= \mathbf{EMod}([0, 1]^X, [0, 1]) = \mathcal{E}(X). \end{aligned}$$

The isomorphism  $\alpha : \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(\mathbb{C}^n, \mathbb{C}) \xrightarrow{\cong} \mathcal{D}(n)$  follows because the expectation and distribution monad coincide on finite sets, see [59]. Explicitly, it is given by  $\alpha(\phi)(i \in n) = \phi(|i\rangle)$  and  $\alpha^{-1}(\varphi)(a) = \sum_i \varphi(i) \cdot a(i)$ . □

In the following, we use  $\theta$  to refer to the map  $\mathbf{BEMod}([0, 1]_A, [0, 1]_B) \rightarrow \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(A, B)$  that exists by Lemma 1.2.11.

**Proposition 1.4.2.** *The expectation monad  $\mathcal{E}(X) \cong \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(\ell^\infty(X), \mathbb{C})$  gives rise to a full and faithful functor:*

$$\begin{array}{ccc} \mathcal{K}\ell(\mathcal{E}) & \xrightarrow{\mathcal{C}_\mathcal{E}} & \mathbf{CC}^*\mathbf{Alg}_{\text{PU}}^{\text{op}} \\ X \vdash & \xrightarrow{\quad} & \ell^\infty(X) \\ (X \xrightarrow{f} \mathcal{E}(Y)) \vdash & \xrightarrow{\quad} & (a \in \ell^\infty(Y) \mapsto (x \in X \mapsto \theta(f(x))(a))). \end{array} \quad (1.5)$$

*Proof.* First, we need to see that  $\mathcal{C}_\mathcal{E}(f)$  is defined – we have to show that the function  $\mathcal{C}_\mathcal{E}(f)(a): X \rightarrow \mathbb{C}$  is bounded. We can apply Lemma 1.2.3 to the function  $f(x) \in \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(\ell^\infty(Y), \mathbb{C})$ ; it yields  $\|\theta(f(x))(a)\| \leq 4\|a\|$ . As this holds for each  $x \in X$ ,  $|\mathcal{C}_\mathcal{E}(f)(a)(x)| = |\theta(f(x))(a)|$  is bounded (by  $4\|a\|$ ) and therefore  $\mathcal{C}_\mathcal{E}(f)(a) \in \ell^\infty(X)$ . Next, the map  $\mathcal{C}_\mathcal{E}(f)$  is a PU-map of  $C^*$ -algebras via the pointwise definitions of the relevant constructions.

We check that  $\mathcal{C}_\mathcal{E}$  preserves (Kleisli) identities and composition. Identities first. Let  $a \in [0, 1]_X$ :

$$\mathcal{C}_\mathcal{E}(\text{id}_X)(a)(x) = \mathcal{C}_\mathcal{E}(\eta_X)(a)(x) = \theta(\eta_X(x))(a) = \eta_X(x)(a) = a(x).$$

So  $\mathcal{C}_\mathcal{E}(\text{id}_X) = \text{id}_{\ell^\infty(X)}$ , because the above holds for all  $a \in \ell^\infty(X)$  by Lemma 1.2.11. For composition, with  $f: X \rightarrow \mathcal{E}(Y)$ ,  $g: Y \rightarrow \mathcal{E}(Z)$ ,  $c \in \ell^\infty(Z)$  and  $x \in X$ :

$$\begin{aligned} \mathcal{C}_\mathcal{E}(g \circ f)(c)(x) &= \theta((g \circ f)(x))(c) \\ &= (g \circ f)(x)(c) \\ &= \mu_Z(\mathcal{E}(g)(f(x)))(c) \\ &= \mathcal{E}(g)(f(x))(\phi \in \mathcal{E}(Z) \mapsto \phi(c)) \\ &= f(x)((\phi \in \mathcal{E}(Z) \mapsto \phi(c)) \circ g) \\ &= f(x)(y \in Y \mapsto g(y)(c)) \\ &= f(x)(\mathcal{C}_\mathcal{E}(g)(c)) \\ &= \mathcal{C}_\mathcal{E}(f)(\mathcal{C}_\mathcal{E}(g)(c))(x) \\ &= (\mathcal{C}_\mathcal{E}(f) \circ \mathcal{C}_\mathcal{E}(g))(c)(x). \end{aligned}$$

By applying Lemma 1.2.11 again, this is so for all  $c \in \ell^\infty(Z)$ , so  $\mathcal{C}_\mathcal{E}(g \circ f) = \mathcal{C}_\mathcal{E}(f) \circ \mathcal{C}_\mathcal{E}(g)$ .

The functor  $\mathcal{C}_{\mathcal{E}}$  is faithful by applying extensionality. To see that  $\mathcal{C}_{\mathcal{E}}$  is full, let  $g : \ell^\infty(Y) \rightarrow \ell^\infty(X)$  be a PU-map. Define  $f : X \rightarrow \mathcal{E}(Y)$  as

$$f(x)(b) = [0, 1]_g(b)(x),$$

where  $x \in X$  and  $b \in [0, 1]^Y$ . We have that  $f(x)$  is the restriction of a PU map  $\ell^\infty(Y) \rightarrow \mathbb{C}$ , so is an effect module map  $\mathbf{EMod}([0, 1]^Y, [0, 1])$ , and therefore an element of  $\mathcal{E}(Y)$ , so  $f$  is a Kleisli morphism. Now, if we take  $b \in [0, 1]^Y$ ,  $x \in X$ , we have

$$\mathcal{C}_{\mathcal{E}}(f)(b)(x) = \theta(f(x))(b) = f(x)(b) = [0, 1]_g(b)(x) = g(b)(x),$$

so  $\mathcal{C}_{\mathcal{E}}(f) = g$  by Lemma 1.2.11.  $\square$

We turn to the finite case, like in the previous section. We do so by considering the Kleisli category  $\mathcal{Kl}_{\mathbb{N}}(\mathcal{E})$  obtained by restricting to objects  $n \in \mathbb{N}$ . Since the expectation monad  $\mathcal{E}$  and the distribution monad  $\mathcal{D}$  coincide on finite sets, we have  $\mathcal{Kl}_{\mathbb{N}}(\mathcal{E}) \cong \mathcal{Kl}_{\mathbb{N}}(\mathcal{D})$ . Maps  $n \rightarrow m$  in this category are probabilistic transition matrices  $n \rightarrow \mathcal{D}(m)$ . This category has been investigated also in [42]. The following equivalence is known, see *e.g.* [80], although possibly not in this categorical form.

**Proposition 1.4.3.** *The functor  $\mathcal{C}_{\mathcal{E}}$  from (1.5) restricts in the finite case to an equivalence of categories:*

$$\mathcal{Kl}_{\mathbb{N}}(\mathcal{D}) \xrightarrow[\simeq]{\mathcal{C}_{\mathcal{D}}} \mathbf{FdCC}^* \mathbf{Alg}_{\mathbf{PU}}^{\text{op}} \quad (1.6)$$

It is given by  $\mathcal{C}_{\mathcal{D}}(n) = \mathbb{C}^n$  and

$$\mathcal{C}_{\mathcal{D}}(n \xrightarrow{f} \mathcal{D}(m))(a \in \mathbb{C}^m)(i \in n) = \sum_{j \in m} f(i)(j) \cdot v(j).$$

This equivalence (1.6) may be read as: the category  $\mathbf{FdCC}^* \mathbf{Alg}_{\mathbf{PU}}$  of finite-dimensional commutative  $\mathbf{C}^*$ -algebras, with positive unital maps, is equivalent to the *Lawvere theory* of the distribution monad  $\mathcal{D}$ .

*Proof.* The fullness and faithfulness of the functor  $\mathcal{C}_{\mathcal{D}}$  follow from Proposition 1.4.2, using the isomorphism  $\mathbf{C}^* \mathbf{Alg}_{\mathbf{PU}}(\mathbb{C}^n, \mathbb{C}) \cong \mathcal{D}(n)$  from Lemma 1.4.1. This functor  $\mathcal{C}_{\mathcal{D}}$  is essentially surjective on objects by Corollary 1.2.5, using the fact that a MIU-map is a PU-map.  $\square$

## 1.5 Continuous Probabilistic Computations

The question arises if the full and faithful functor  $\mathcal{Kl}(\mathcal{E}) \rightarrow \mathbf{CC}^*\mathbf{Alg}_{\mathbf{PU}}^{\text{op}}$  from Proposition 1.4.2 can be turned into an equivalence of categories, but not just for the finite case like in Proposition 1.4.3. In order to make this work we have to lift the expectation monad  $\mathcal{E}$  on **Set** to the category **CHaus** of compact Hausdorff spaces. For this purpose we use what we call the *Radon* monad  $\mathcal{R}$ , defined on  $X \in \mathbf{CHaus}$  as:

$$\mathcal{R}(X) = \text{Stat}(C(X)) = \mathbf{C}^*\mathbf{Alg}_{\mathbf{PU}}(C(X), \mathbb{C}), \quad (1.7)$$

where, as usual,  $C(X) = \{f: X \rightarrow \mathbb{C} \mid f \text{ is continuous}\}$ ; notice that the functions  $f \in C(X)$  are automatically bounded, since  $X$  is compact. We have implicitly applied the forgetful functor from **CCL**  $\rightarrow$  **CHaus** to make  $\mathcal{R}$  into an endofunctor of **CHaus**. The elements of  $\mathcal{R}(X)$  are related to measures in the following way. If  $\nu$  is a probability measure on the Borel sets of  $X$ , integration of continuous functions with respect to  $\nu$  gives a function  $a \mapsto \int_X a \, d\nu \in \mathcal{R}(X)$ . A Radon probability measure, or an inner regular probability measure, is one such that  $\nu(S) = \sup_{K \subseteq S} \nu(K)$  where  $K$  ranges over compact sets. The map from measures to elements of  $\mathcal{R}(X)$  is a bijection [107, Thm. 2.14], and accordingly we shall sometimes refer to elements of  $\mathcal{R}(X)$  as measures. Therefore the Radon monad can be considered to be a variant of the Giry monad. We explain this more precisely in Section 1.6. The Radon monad differs from the monad Giry defined on the category of Polish spaces essentially only in the choice of spaces, and on compact Polish spaces they agree, as the topology Giry used is the same as the weak-\* topology, and Polish spaces do not admit any non-Radon Borel probability measures [13, Theorems 1.1 and 1.4]. There are, however, non-Radon Borel probability measures on some unmetrizable compact Hausdorff spaces [41, 434K (d), page 192] [49, §53.10, page 231].

This Radon monad  $\mathcal{R}$  is not new: we shall see later that it occurs in [116, Theorem 3] as the monad of an adjunction (“probability measure” is used to mean “Radon probability measure” in that article). It is proven to be a monad independently of that paper in [37, Theorem 2.13], and it has been used more recently in [85]. However, our duality result below — Theorem 1.5.1 — is not known in the literature, at least in this form. We discuss, in Section 1.6 how it relates to a theorem of Umegaki.

From Proposition 1.2.6 it is immediate that  $\mathcal{R}(X)$  is again a compact Hausdorff space. On continuous maps  $f: X \rightarrow Y$  is defined as

$$\mathcal{R}(f)(\phi)(b) = \phi(b \circ f),$$

where  $\phi \in \mathcal{R}(X)$  and  $b \in C(Y)$ . The unit  $\eta_X : X \rightarrow \mathcal{R}(X)$  and multiplication  $\mu_X : \mathcal{R}^2(X) \rightarrow \mathcal{R}(X)$  are defined similarly to the expectation monad, namely as  $\eta(x)(a) = a(x)$  and  $\mu(\Phi)(a) = \Phi(\psi \mapsto \psi(a))$ . We check that  $\eta_X$  is continuous. Recall from the proof of Proposition 1.2.6 that a basic open in  $\mathcal{R}(X)$  is of the form  $\text{ev}_s^{-1}(U) = \{h \in \mathcal{R}(X) \mid h(s) \in U\}$ , where  $s \in C(X)$  and  $U \subseteq \mathbb{C}$  is open. Then:

$$\eta_X^{-1}(\text{ev}_s^{-1}(U)) = \{x \in X \mid \eta_X(x)(s) \in U\} = \{x \in X \mid s(x) \in U\} = s^{-1}(U).$$

The latter is an open subset of  $X$  since  $s : X \rightarrow \mathbb{C}$  is a continuous function.

We are now ready to state our main, new duality result. It may be understood as a probabilistic version of Gelfand duality, for commutative  $C^*$ -algebras with PU maps instead of the MIU maps originally used (see Theorem 1.2.4).

**Theorem 1.5.1.** *In the case of the Radon monad (1.7) there is an equivalence of categories:*

$$\mathcal{Kl}(\mathcal{R}) \simeq \mathbf{CC}^*\mathbf{Alg}_{\text{PU}}^{\text{op}}.$$

*Proof.* We define a functor  $\mathcal{C}_{\mathcal{R}} : \mathcal{Kl}(\mathcal{R}) \rightarrow \mathbf{CC}^*\mathbf{Alg}_{\text{PU}}^{\text{op}}$  like in (1.5), namely by:

$$\mathcal{C}_{\mathcal{R}}(X) = C(X) \qquad \mathcal{C}_{\mathcal{R}}(f)(b)(x) = f(x)(b).$$

We must first show that  $\mathcal{C}_{\mathcal{R}}(f)(b) \in C(X)$ . If  $(x_i)_{i \in I}$  is a net converging to a point  $x \in X$ , we want to show  $\mathcal{C}_{\mathcal{R}}(f)(b)(x_i) \rightarrow \mathcal{C}_{\mathcal{R}}(f)(b)(x)$ . We have

$$\mathcal{C}_{\mathcal{R}}(f)(b)(x_i) = f(x_i)(b).$$

As  $f$  is continuous,  $f(x_i) \rightarrow f(x)$ , and as evaluating at  $b$  is continuous in the weak-\* topology, we have

$$f(x_i)(b) \rightarrow f(x)(b) = \mathcal{C}_{\mathcal{R}}(f)(b)(x).$$

So  $\mathcal{C}_{\mathcal{R}}(f)(b)$  is continuous, and therefore an element of  $C(X)$ .

As in the  $\ell^\infty$  case, pointwiseness of the operations implies that  $\mathcal{C}_{\mathcal{R}}(f)$  is a PU-map. The proof that  $\mathcal{C}_{\mathcal{R}}$  is a functor is similar to the proof in Proposition 1.4.2. The proof that  $\mathcal{C}_{\mathcal{R}}$  is faithful is by functional extensionality and is immediate.

We show that  $\mathcal{C}_{\mathcal{R}}$  is full as follows. Let  $g : C(Y) \rightarrow C(X)$  be a PU-map. Define  $f : X \rightarrow \mathcal{R}(Y)$  as

$$f(x)(b) = g(b)(x),$$

where  $x \in X$  and  $b \in C(Y)$ . We have that  $f(x) \in \mathcal{R}(Y)$  by the pointwiseness of the operations. We show that  $f$  is continuous from  $X$  to the weak-\* topology

on  $\mathcal{R}(Y)$  as follows. Let  $(x_i)_{i \in I}$  be a net converging to  $x \in X$ . For all  $b \in C(Y)$ , we have  $f(x_i)(b) = g(b)(x_i)$ . As  $g(b) \in C(X)$ , we have  $g(b)(x_i) \rightarrow g(b)(x) = f(x)(b)$ . As this is so for all  $b \in C(Y)$ , we have  $f(x_i) \rightarrow f(x)$  in the weak- $*$  topology.

We have shown that  $f$  is a Kleisli map, so we only need to show that  $\mathcal{C}_{\mathcal{R}}(f) = g$  to show fullness. We have

$$\mathcal{C}_{\mathcal{R}}(f)(b)(x) = f(x)(b) = g(b)(x).$$

The functor is essentially surjective on objects by ordinary Gelfand duality (Theorem 1.2.4), because  $*$ -homomorphisms are also PU-maps.  $\square$

As suggested by a member of the thesis committee, we remark at this point that it is possible to generalize the previous proof by varying the kind of map  $M$  we use on commutative  $C^*$ -algebras, and taking the relevant monad to have  $\mathbf{C}^*\mathbf{Alg}_M(C(X), \mathbb{C})$  as its underlying functor. In particular, using positive *subunital* maps, we would obtain a duality for the Kleisli category of the subprobabilistic Radon monad (proven to be a monad in [68, §7]). This cannot be pushed arbitrarily far, however, as if we chose bounded maps,  $\mathbf{C}^*\mathbf{Alg}(C(X), \mathbb{C})$  would not be compact in the weak- $*$  topology except for the trivial case where  $X = \emptyset$ .

We investigate the Radon monad  $\mathcal{R}$  a bit further, in particular its relation to the distribution monad  $\mathcal{D}$  on **Set**.

**Lemma 1.5.2.** *There is a lax map of monads  $(U, \tau): \mathcal{R} \rightarrow \mathcal{D}$  in:*

$$\begin{array}{ccc} \begin{array}{c} \mathcal{R} \\ \curvearrowright \\ \mathbf{CHaus} \end{array} & \xrightarrow{U} & \begin{array}{c} \mathbf{Set} \\ \curvearrowright^{\mathcal{D}} \end{array} \end{array} \quad \mathcal{D}U \xRightarrow{\tau} U\mathcal{R}$$

where  $U$  is the forgetful functor and  $\tau$  commutes appropriately with the units and multiplications of the monads  $\mathcal{D}$  and  $\mathcal{R}$ . (Such a map is called a “monad functor” in [115, §1].)

As a result, the forgetful functor can be lifted to the associated categories of Eilenberg-Moore algebras:

$$\begin{array}{ccc} \mathcal{EM}(\mathcal{R}) & \xrightarrow{\quad} & \mathcal{EM}(\mathcal{D}) \\ (\mathcal{R}(X) \xrightarrow{\alpha} X) & \mapsto & (\mathcal{D}(UX) \xrightarrow{\tau} U\mathcal{R}(X) \xrightarrow{U\alpha} UX) \end{array}$$

Therefore the carrier of an  $\mathcal{R}$ -algebra is a convex compact Hausdorff space, and every algebra map is an affine function.



*Proof.* For  $X \in \mathbf{CHaus}$  and  $\varphi \in \mathcal{D}(UX)$ , that is for  $\varphi: UX \rightarrow [0, 1]$  with finite support and  $\sum_{x \in X} \varphi(x) = 1$ , we define  $\tau_X(\varphi) \in U\mathcal{R}(X)$  on  $a \in C(X)$  as:

$$\tau_X(\varphi)(a) = \sum_{x \in X} \varphi(x) \cdot a(x) \in \mathbb{C}. \quad (1.8)$$

It is easy to see that  $\tau$  is a linear map  $C(X) \rightarrow \mathbb{C}$  that preserves positive elements and the unit. Moreover, it commutes appropriately with the units and multiplications. For instance:

$$(\tau_X \circ \eta_{UX}^{\mathcal{D}})(x)(a) = \tau_X(\delta_x)(a) = a(x) = U(\eta_X^{\mathcal{R}})(x)(a).$$

□

The continuous dual space of  $C(X)$  can be ordered using (0.1), by taking the positive cone to be those linear functionals that map positive functions to positive numbers (the dual cone of  $C(X)_+$ , see Lemma 0.3.8).

**Definition 1.5.3.** A state  $\phi \in \mathcal{R}(X) = \mathbf{C}^*\mathbf{Alg}_{\text{PU}}(C(X), \mathbb{C})$  is a pure state if for each positive linear functional such that  $\psi \leq \phi$ , i.e. such that  $\phi - \psi$  is positive, there exists an  $\alpha \in [0, 1]$  such that  $\psi = \alpha\phi$ . □

**Lemma 1.5.4.** For a compact Hausdorff space  $X$ , the subset of Dirac measures  $\{\eta(x) \mid x \in X\} \subseteq \mathcal{R}(X)$  is exactly the set of pure states and therefore the set of extreme points of the set of Radon measures  $\mathcal{R}(X)$  – where  $\eta(x) = \eta^{\mathcal{R}}(x)$  is the unit of the monad  $\mathcal{R}$ .

*Proof.* We rely on the basic fact, see [26, 2.5.2, page 43], that a measure is a Dirac measure iff it is a pure state. We prove the above lemma by showing that the pure states are precisely the extreme points of the convex set  $\mathcal{R}(X)$ .

- If  $\phi \in \mathcal{R}(X)$  is a pure state, suppose  $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$ , a convex combination of two states  $\phi_i \in \mathcal{R}(X)$  with  $\alpha_i \in [0, 1]$  satisfying  $\alpha_1 + \alpha_2 = 1$ , where no two elements of  $\{\phi, \phi_1, \phi_2\}$  are the same. Then  $\phi \geq \alpha_1\phi_1$ , since for a positive function  $f \in C(X)$  one has  $(\phi - \alpha_1\phi_1)(f) = \alpha_2\phi_2(f) \geq 0$ . Thus  $\alpha_1\phi_1 = \alpha\phi$ , for some  $\alpha \in [0, 1]$ , since  $\phi$  is pure. Then  $\alpha_1 = \alpha_1\phi_1(1) = \alpha\phi(1) = \alpha$ . If  $\alpha_1 = 0$ , then  $\alpha_2 = 1$  and so  $\phi = \phi_2$ . If  $\alpha_1 > 0$ , then  $\phi = \phi_1$ . Hence  $\phi$  is an extreme point.
- Suppose  $\phi$  is an extreme point of  $\mathcal{R}(X)$ , i.e. that  $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$  implies  $\phi_1$  or  $\phi_2 = \phi$ . Then if there is a positive linear functional  $\psi \leq \phi$ , we may take  $\alpha_1 = \psi(1) \geq 0$ ; since  $\alpha_1 = \psi(1) \leq \phi(1) = 1$ , we get  $\alpha_1 \in [0, 1]$ .

If  $\alpha_1 = 0$ , then since  $\|\psi\| = \psi(1) = 0$  we get  $\psi = 0$  and  $\psi = 0 \cdot \phi$ . If  $\alpha_1 = 1$ , then  $(\phi - \psi)(1) = 0$ , which since  $\phi - \psi$  was assumed to be positive implies  $\phi - \psi = 0$  and hence  $\psi = 1 \cdot \phi$ . Having dealt with those cases, we have that  $\alpha_1 \in (0, 1)$ , and so we have a state  $\phi_1 = \frac{1}{\alpha_1}\psi$ . We may take  $\alpha_2 = 1 - \alpha_1 \in (0, 1)$  and obtain a second state  $\phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$ . By construction we have a convex decomposition of  $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$ . Therefore either  $\phi = \phi_1 = \frac{1}{\alpha_1}\psi$  or  $\phi = \phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$ . In the first case,  $\psi = \alpha_1\phi$ , making  $\phi$  pure. But also in the second case  $\phi$  is pure, since we have  $\alpha_2\phi = \phi - \psi$  and thus  $\psi = (1 - \alpha_2)\phi$ .  $\square$

**Lemma 1.5.5.** *Let  $X$  be a compact Hausdorff space.*

- (i) *The maps  $\tau_X: \mathcal{D}(UX) \rightarrow U\mathcal{R}(X)$  from (1.8) are injective; as a result, the unit/Dirac maps  $\eta: X \rightarrow \mathcal{R}(X)$  are also injective.*
- (ii) *The maps  $\tau_X: \mathcal{D}(UX) \rightarrow U\mathcal{R}(X)$  embed  $\mathcal{D}(UX)$  as a dense subset of  $U\mathcal{R}(X)$ .*

*Proof.* For the first point, assume  $\varphi, \psi \in \mathcal{D}(UX)$  satisfying  $\tau(\varphi) = \tau(\psi)$ . We first show that the finite support sets are equal:  $\text{supp}(\varphi) = \text{supp}(\psi)$ . Since  $X$  is Hausdorff, singletons are closed, so finite subsets are too. Suppose  $\text{supp}(\varphi) \not\subseteq \text{supp}(\psi)$ , so that  $S = \text{supp}(\varphi) - \text{supp}(\psi)$  is non-empty. Since  $S$  and  $\text{supp}(\psi)$  are disjoint closed subsets, there is, by Urysohn's lemma, a continuous function  $f: X \rightarrow [0, 1]$  with  $f(x) = 1$  for  $x \in S$  and  $f(x) = 0$  for  $x \in \text{supp}(\psi)$ . But then  $\tau(\psi)(f) = 0$ , whereas  $\tau(\varphi)(f) \neq 0$ .

Now that we know  $\text{supp}(\varphi) = \text{supp}(\psi)$ , assume  $\varphi(x) \neq \psi(x)$ , for some  $x \in \text{supp}(\varphi)$ . The closed subsets  $\{x\}$  and  $\text{supp}(\varphi) - \{x\}$  are disjoint, so there is, again by Urysohn's lemma, a continuous function  $f: X \rightarrow [0, 1]$  with  $f(x) = 1$  and  $f(y) = 0$  for all  $y \in \text{supp}(\varphi)$ . But then  $\varphi(x) = \tau(\varphi)(f) = \tau(\psi)(f) = \psi(x)$ , contradicting the assumption.

We can conclude that the unit  $X \rightarrow \mathcal{R}(X)$  is also injective, since its underlying function can be written as the composite  $U(\eta^{\mathcal{R}}) = \tau \circ \eta^{\mathcal{D}}: UX \rightarrow \mathcal{D}(UX) \rightarrow U\mathcal{R}(X)$ , because  $\tau$  is a lax map of monads.

To show that the image of  $\tau_X$  is dense, we proceed as follows. By Lemmas 1.5.4 and 1.5.2, the extreme points of  $\mathcal{R}(X)$  are

$$\{\eta^{\mathcal{R}}(x) \mid x \in X\} = \{\tau(\eta^{\mathcal{D}}(x)) \mid x \in X\}$$

and are thus in the image of  $\tau: \mathcal{D}(UX) \rightarrow U\mathcal{R}(X)$ . Since every convex combination of  $\eta^{\mathcal{R}}(x)$  comes from a formal convex sum  $\varphi \in \mathcal{D}(UX)$ , all convex combinations of extreme points are in the image of  $\tau_X$ . Using Proposition

1.2.6,  $\mathcal{R}(X)$  can be considered an object of **CCL**, i.e. a compact convex subset of a locally convex space. Accordingly, we may apply the Krein-Milman theorem [20, Proposition 7.4, page 142] to conclude the set of convex combinations of extreme points is dense.  $\square$

**Lemma 1.5.6.** *Let  $X, Y$  be compact Hausdorff spaces. The structure map of each Eilenberg-Moore algebra  $\alpha: \mathcal{R}(X) \rightarrow X$  is a  $\mathcal{D}$ -affine function. For each continuous map  $f: X \rightarrow Y$ , the function  $\mathcal{R}(f): \mathcal{R}(X) \rightarrow \mathcal{R}(Y)$  is  $\mathcal{D}$ -affine.*

*Proof.* This follows from the naturality of  $\tau: \mathcal{D}U \Rightarrow UR$ .  $\square$

**Proposition 1.5.7.** *Let  $\alpha: \mathcal{R}(X) \rightarrow X$  and  $\beta: \mathcal{R}(Y) \rightarrow Y$  be two Eilenberg-Moore algebras of the Radon monad  $\mathcal{R}$ . A function  $f: X \rightarrow Y$  is an algebra homomorphism if and only if  $f$  is both continuous and affine.*

*As a result, the functor  $\mathcal{EM}(\mathcal{R}) \rightarrow \mathcal{EM}(\mathcal{D})$  from Lemma 1.5.2 is faithful, and an  $\mathcal{EM}(\mathcal{D})$  map comes from an  $\mathcal{EM}(\mathcal{R})$  map if and only if it is continuous.*

We shall follow the convention of writing  $\text{CAff}(X, Y)$  for the homset of continuous and  $\mathcal{D}$ -affine functions  $X \rightarrow Y$ .

*Proof.* Clearly, each algebra map is both continuous and  $\mathcal{D}$ -affine. For the converse, if  $f: X \rightarrow Y$  is continuous, it is a map in the category **CHaus** of compact Hausdorff spaces. Since it is  $\mathcal{D}$ -affine, both triangles commute in:

$$\begin{array}{ccc} \mathcal{D}(UX) & \xrightarrow[\text{dense}]{\tau} & \mathcal{R}(X) \\ & \searrow & \downarrow f \circ \alpha \quad \downarrow \beta \circ \mathcal{R}(f) \\ & & Y \end{array}$$

Since  $Y$  is Hausdorff, there is at most one such map. Therefore  $f$  is an algebra map.  $\square$

The category  $\mathcal{EM}(\mathcal{R})$  of Eilenberg-Moore algebras of the Radon monad may thus be understood as a category of convex compact Hausdorff spaces, with affine continuous maps between them. In Chapter 4 we see how to use a result from [116] to relate this to **CCL**, which is a category of “concrete” convex sets. Using this theorem, it will be shown that “observability” conditions like in [59, top of p. 169] always hold for algebras of  $\mathcal{R}$ .

In the case of the expectation monad  $\mathcal{E}$ , it is not necessary to use a forgetful functor to relate it to  $\mathcal{D}$  as they are both defined on the same category, **Set**.

There is a monad morphism  $\sigma : \mathcal{D} \Rightarrow \mathcal{E}$  defined, for  $X \in \mathbf{Set}$ ,  $\phi \in \mathcal{D}(X)$ , and  $a \in [0, 1]^X$  as

$$\sigma_X(\phi)(a) = \sum_{x \in X} \phi(x) \cdot a(x).$$

The proof of this is given in [59, Lemma 21]. There is also a monad morphism  $\tau : \mathcal{U} \Rightarrow \mathcal{E}$  defined as follows, with  $\mathcal{F} \in U(X)$  and  $a \in [0, 1]^X$ :

$$\tau_X(\mathcal{F})(a) = \text{ch}(\mathcal{U}(a)(\mathcal{F})),$$

where  $\text{ch}$  is the unit interval's  $\mathcal{EM}(\mathcal{U})$  structure arising from its being a compact Hausdorff space in its usual topology, as described in Example 0.4.12. The proof that this is a monad morphism is detailed in [59, Proposition 16].

For later reference, we summarize these results as follows

**Proposition 1.5.8.** *There exist monad morphisms  $\tau : \mathcal{U} \Rightarrow \mathcal{E}$  and  $\sigma : \mathcal{D} \Rightarrow \mathcal{E}$ . These induce forgetful functors  $\mathcal{EM}(\mathcal{E}) \rightarrow \mathcal{EM}(\mathcal{U}) \simeq \mathbf{CHaus}$ , showing that every  $\mathcal{E}$ -algebra is canonically a compact Hausdorff space and every map of  $\mathcal{E}$ -algebras is continuous, and  $\mathcal{EM}(\mathcal{E}) \rightarrow \mathcal{EM}(\mathcal{D})$ , showing that every  $\mathcal{E}$ -algebra is canonically an abstract convex set, and every map of  $\mathcal{E}$ -algebras is affine.  $\square$*

In Chapter 4 we will see that  $\mathcal{EM}(\mathcal{E})$  is in fact equivalent to  $\mathbf{CCL}$ .

## 1.6 The Category of Markov Kernels

In this section we relate the rest of the chapter to the literature on Markov kernels and the Giry monad. First, we recall some notions from measure theory. Recall that a *measurable space* is a pair  $(X, \Sigma)$  where  $X$  is a set and  $\Sigma$  a  $\sigma$ -algebra on  $X$ , *i.e.* a family of subsets of  $X$  closed under countable intersections, unions and complements. A function  $f : (X, \Sigma) \rightarrow (Y, \Theta)$  between measurable spaces is called a *measurable map* if for all  $T \in \Theta$ ,  $f^{-1}(T) \in \Sigma$ . This is analogous to the definition of a continuous function between topological spaces. Measurable maps form a category,  $\mathbf{Mes}$ .

On any topological space  $X$ , we can define a measure space structure by taking the  $\sigma$ -algebra generated by the open sets. This is known as the *Borel  $\sigma$ -algebra*,  $\mathcal{Bo}(X)$ , and its elements are called *Borel sets*. A *measurable function* on  $(X, \Sigma)$  is a measurable map  $a : (X, \Sigma) \rightarrow (\mathbb{C}, \mathcal{Bo}(\mathbb{C}))$ . The set of bounded measurable functions on  $(X, \Sigma)$ ,  $\mathcal{L}^\infty(X, \Sigma)$ , is a closed  $*$ -subalgebra of  $\ell^\infty(X)$ .

If  $(X, \Sigma)$  and  $(Y, \Theta)$  are measurable spaces, then a *Markov kernel* is a function  $f : X \times \Theta \rightarrow [0, 1]$  such that

- (i) For all  $x \in X$ ,  $f(x, -) : \Theta \rightarrow [0, 1]$  is a probability measure.

- (ii) For all  $T \in \Theta$ ,  $f(-, T) : X \rightarrow [0, 1]$  is a measurable function ( $[0, 1]$  being equipped with the Borel  $\sigma$ -algebra).

Composition of Markov kernels  $f : X \times \Theta \rightarrow [0, 1]$  and  $g : Y \times \Xi \rightarrow [0, 1]$ , for measurable spaces  $(X, \Sigma)$ ,  $(Y, \Theta)$  and  $(Z, \Xi)$  is defined by the formula

$$(g * f)(x, U) = \int_Y g(-, U) \, df(x, -), \quad (1.9)$$

*i.e.* we integrate the measurable function  $g(-, U)$  with respect to the measure  $f(x, -)$ . This is often written using a dummy variable  $y$  as

$$(g * f)(x, U) = \int_Y f(x, dy) g(y, U)$$

An identity operation for this composition is defined by Dirac measures

$$\text{id}_X(x, S) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}. \quad (1.10)$$

The category of Markov kernels, **Markov**, is defined to have measurable spaces as objects, and Markov kernels as maps. It was originally defined in [78] and independently in [121, Chapter 5]. However, Kolmogorov gave a definition of the endomorphisms of this category in [72], including proofs of the identity laws and associativity of composition. The composition law (1.9) is therefore known as the Chapman-Kolmogorov equation in subsequent work on stochastic processes [38] [27, §VI.2]. A proof that it is a category, in the subprobabilistic case, can be found in [92, Chapter 5], where it is known as the category of stochastic relations.

The Giry monad  $\mathcal{G}$  is a monad on the category of measurable spaces and measurable maps,  $\mathcal{Mes}$ , defined as follows. For a measurable space  $(X, \Sigma)$ ,

$$\mathcal{G}(X, \Sigma) = \{\nu : \Sigma \rightarrow [0, 1] \mid \nu \text{ a probability measure}\}.$$

For each  $S \in \Sigma$ , a map  $p_S : \mathcal{G}(X) \rightarrow [0, 1]$  is defined as

$$p_S(\nu) = \nu(S).$$

The  $\sigma$ -algebra on  $\mathcal{G}(X)$  is defined to be the coarsest  $\sigma$ -algebra such that  $p_S$  is measurable, for all  $S \in \Sigma$ . Equivalently, this can be described as the  $\sigma$ -algebra on  $\mathcal{G}(X)$  generated by the sets of the form  $p_S^{-1}(B)$  where  $B$  is a Borel subset of  $[0, 1]$ . This defines  $\mathcal{G}$  on objects. On a measurable map  $f : (X, \Sigma) \rightarrow (Y, \Theta)$

$$\mathcal{G}(f)(\nu)(T) = \nu(f^{-1}(T)),$$

where  $\nu \in \mathcal{G}(X)$  and  $T \in \Theta$ . This defines  $\mathcal{G}$  as a functor  $\mathcal{Mes} \rightarrow \mathcal{Mes}$ . The unit is defined as

$$\eta_X : (X, \Sigma) \rightarrow \mathcal{G}(X, \Sigma)$$

$$\eta_X(x)(S) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases},$$

where  $S \in \Sigma$ . The multiplication is defined as

$$\mu_X : \mathcal{G}^2(X, \Sigma) \rightarrow \mathcal{G}(X, \Sigma)$$

$$\mu_X(\Phi)(S) = \int_{\mathcal{G}(X)} p_S d\Phi. \quad (1.11)$$

We will now give proofs of some standard useful facts about the Giry monad. First, we can extend the definition of  $p_-$  from measurable subsets to measurable functions, as follows. If  $a \in \mathcal{L}^\infty(X, \Sigma)$ , i.e.  $a$  is a bounded measurable  $\mathbb{R}$ -valued function, we define

$$p_a(\nu) = \int_X a d\nu.$$

It is clear that  $p_{\chi_S} = p_S$  for all  $S \in \Sigma$ , so this extends the original definition.

**Lemma 1.6.1.** *For any measurable space  $(X, \Sigma)$  and  $a \in \mathcal{L}^\infty(X, \Sigma)$ ,  $p_a \in \mathcal{L}^\infty(\mathcal{G}(X))$ . The mapping  $p_-$  is linear, and if  $(a_i)_{i \in \mathbb{N}}$  is a bounded sequence in  $\mathcal{L}^\infty(X, \Sigma)$  that converges pointwise to  $a \in \mathcal{L}^\infty(X, \Sigma)$ , we have  $p_{a_i} \rightarrow p_a$  pointwise on  $\mathcal{G}(X)$ .*

*Proof.* We first show that  $p_a$  is bounded, and therefore  $p_a \in \ell^\infty(\mathcal{G}(X))$ . As  $a$  is bounded, there exists a number  $\alpha \in \mathbb{R}_{\geq 0}$  such that  $-\alpha \leq a \leq \alpha$ . As integration  $\int_X \cdot d\nu$  is a linear, positive and unital map  $\mathcal{L}^\infty(X) \rightarrow \mathbb{R}$  for all  $\nu \in \mathcal{G}(X)$ , we have

$$-\alpha = \int_X -\alpha d\nu \leq \int_X a d\nu \leq \int_X \alpha d\nu = \alpha.$$

Therefore  $p_a$  is bounded by  $\alpha$  as  $\nu$  varies over  $\mathcal{G}(X)$ .

To show that  $p_a$  is measurable, we will use the other two facts, so we prove them first. If we consider a linear combination in  $\mathcal{L}^\infty(X)$  of the form  $\alpha a + \beta b$ , we have

$$p_{\alpha a + \beta b}(\nu) = \int_X (\alpha a + \beta b) d\nu = \alpha \int_X a d\nu + \beta \int_X b d\nu = (\alpha p_a + \beta p_b)(\nu)$$

Therefore  $p_- : \mathcal{L}^\infty(X) \rightarrow \ell^\infty(\mathcal{G}(X))$  is linear.

Now suppose that  $(a_i)_{i \in \mathbb{N}}$  is a bounded sequence in  $\mathcal{L}^\infty(X)$  converging pointwise to a bounded measurable function  $a$ . Then

$$p_a(\nu) = \int_X a \, d\nu = \int_X \lim_{i \rightarrow \infty} a_i \, d\nu = \lim_{i \rightarrow \infty} \int_X a_i \, d\nu = \lim_{i \rightarrow \infty} p_{a_i}$$

by the dominated convergence theorem [111, Theorem 11.2]. Therefore  $p_{a_i}$  converges pointwise to  $p_a$ .

We can now show that  $p_a$  is a measurable function with respect to the usual  $\sigma$ -algebra on  $\mathcal{G}(X)$ . If  $a = \chi_S$  for some measurable set  $S$ , then  $p_{\chi_S} = p_S$  and is therefore measurable. As measurable functions are closed under linear combinations, we therefore have that if  $a$  is a simple function,  $p_a$  is measurable (using linearity of  $p_-$ ). Now, for any bounded measurable function  $a$ , there exists a bounded sequence of simple functions  $(a_i)_{i \in \mathbb{N}}$  such that  $a_i \rightarrow a$  pointwise [111, Theorem 8.8]. Therefore  $p_a = \lim_{i \rightarrow \infty} p_{a_i}$  is the pointwise limit of a sequence of measurable functions, and therefore is measurable [111, Corollary 8.9].  $\square$

As a consequence, the  $\sigma$ -algebra on  $\mathcal{G}(X)$  could equally well have been defined as the coarsest such that each  $p_a$  is measurable, as  $a$  varies over  $\mathcal{L}^\infty(X)$ .

The Giry monad's functor, unit and multiplication are defined in terms of measurable subsets. One can use the dominated convergence theorem to re-express these in terms of integration of bounded measurable functions. As the proofs are similar, we do them all at once.

**Proposition 1.6.2.**

- (i) If  $f : (X, \Sigma) \rightarrow (Y, \Theta)$  is a measurable map,  $\nu \in \mathcal{G}(X, \Sigma)$  and  $b \in \mathcal{L}^\infty(Y, \Theta)$ :

$$\int_Y b \, d\mathcal{G}(f)(\nu) = \int_X b \circ f \, d\nu.$$

- (ii) For any measurable space  $(X, \Sigma)$ ,  $a \in \mathcal{L}^\infty(X, \Sigma)$  and  $x \in X$ :

$$\int_X a \, d\eta_X(x) = a(x).$$

- (iii) For any measurable space  $(X, \Sigma)$ ,  $\Phi \in \mathcal{G}^2(X, \Sigma)$  and  $a \in \mathcal{L}^\infty(X, \Sigma)$ :

$$\int_X a \, d\mu_X(\Phi) = \int_{\mathcal{G}(X)} p_a \, d\Phi.$$

*Proof.* In each case we show that the equation holds for characteristic functions of measurable subsets and that certain parts are linear and preserve pointwise convergent sequences, as needed. In each case, it then follows from the fact that every measurable bounded function is a pointwise limit of a sequence of simple functions [111, Theorem 8.8] and the dominated convergence theorem [111, Theorem 11.2].

- (i) We show it on characteristic functions of measurable sets as follows. Let  $T \in \Theta$ :

$$\begin{aligned} \int_Y \chi_T d\mathcal{G}(f)(\nu) &= \mathcal{G}(f)(\nu)(T) = \nu(f^{-1}(T)) = \int_X \chi_{f^{-1}(T)} d\nu \\ &= \int_X \chi_T \circ f d\nu. \end{aligned}$$

We can then observe that  $- \circ f$  is linear and preserves pointwise convergence by evaluating at an arbitrary point  $x \in X$  and using the pointwiseness of the definitions.

- (ii) We show it on characteristic functions as follows. Let  $S \in \Sigma$ :

$$\int_X \chi_S d\eta_X(x) = \eta_X(x)(S) = \chi_S(x),$$

the last equality being shown by reasoning by cases. We have that evaluating at a point is linear and preserves all pointwise limits (not just sequential), so we can conclude that it holds for all measurable functions.

- (iii) We show it on characteristic functions as follows. Let  $S \in \Sigma$ :

$$\int_X \chi_S d\mu_X(\Phi) = \mu_X(\Phi)(S) = \int_{\mathcal{G}(X)} p_S d\Phi = \int_{\mathcal{G}(X)} p_{\chi_S} d\Phi.$$

We can then use the fact that  $p_-$  is linear and preserves pointwise sequential limits from Lemma 1.6.1.  $\square$

The above facts are standard (for instance (i) is also proven in [106, §15.1 Proposition 1] and [92, Proposition 3.8 and p. 66]), we give a proof only for convenience of the reader.

In a similar spirit, we now give a proof that  $\mathcal{K}\ell(\mathcal{G})$  is equivalent to **Markov**. This can also be found, in the subprobabilistic case, in [92, p. 69]. We define



$F : \mathcal{Kl}(G) \rightarrow \mathbf{Markov}$  to be the identity on objects, and on a Kleisli morphism  $f : (X, \Sigma) \rightarrow (Y, \Theta)$  (i.e.  $f : (X, \Sigma) \rightarrow \mathcal{G}(Y, \Theta)$  as a measurable map)

$$F(f)(x, T) = f(x)(T),$$

where  $x \in X$  and  $T \in \Theta$ .

**Proposition 1.6.3.**  *$F$  is a functor  $\mathcal{Kl}(\mathcal{G}) \rightarrow \mathbf{Markov}$ . It is a bijection on morphisms, and therefore an isomorphism of categories.*

*Proof.* Let  $f : (X, \Sigma) \rightarrow [0, 1]^\Theta$  be a function (e.g. if  $f$  is a Kleisli map). As  $F(f)(x, -) = f(x)$ , we have that  $F(f)(x, -)$  is a measure iff  $f(x)$  is. If  $T \in \Theta$  we can see that

$$(p_T \circ f)(x) = p_T(f(x)) = f(x)(T) = F(f)(x, T),$$

so  $p_T \circ f = F(f)(-, T)$ . We therefore have that  $f$  is measurable with respect to the Giry  $\sigma$ -algebra iff  $F(f)(-, T)$  is measurable for all  $T \in \Theta$ . Therefore a map  $f : (X, \Sigma) \rightarrow [0, 1]^\Theta$  is a measurable map  $(X, \Sigma) \rightarrow \mathcal{G}(Y, \Theta)$  iff  $F(f)$  is a Markov kernel.

To show that  $F$  is a functor, we need to show that it preserves identities and composition. We have, for a measurable space  $(X, \Sigma)$  and  $x \in X$ ,  $S \in \Sigma$ :

$$F(\text{id}_X)(x, S) = F(\eta_X)(x, S) = \eta_X(x)(S) = \text{id}_X(x, S),$$

where the last equation is by case-by-case reasoning using (1.10).

Now let  $f : (X, \Sigma) \rightarrow \mathcal{G}(Y, \Theta)$  and  $g : (Y, \Theta) \rightarrow \mathcal{G}(Z, \Xi)$  be maps in  $\mathcal{Kl}(\mathcal{G})$ . Their Kleisli composition is  $\mu_Z \circ \mathcal{G}(g) \circ f$ , so we want to show that  $F(\mu_Z \circ \mathcal{G}(g) \circ f) = F(g) * F(f)$ . Let  $x \in X$  and  $U \in \Xi$ . Then

$$\begin{aligned} F(\mu_Z \circ \mathcal{G}(g) \circ f)(x, U) &= \mu_Z(\mathcal{G}(g)(f(x)))(U) && \text{definition of } F \\ &= \int_{\mathcal{G}(Z)} p_U \, d\mathcal{G}(g)(f(x)) && (1.11) \\ &= \int_Y p_U \circ g \, df(x) && \text{Proposition 1.6.2 (i)} \\ &= \int_Y F(g)(-, U) \, dF(f)(x, -) \\ &= (F(g) * F(f))(x, U) && (1.9). \end{aligned}$$

As  $F$  is the identity on objects, to prove that  $F$  is an isomorphism of categories, we only need to prove that it is bijective on hom-sets. This is because the functor defined as the identity on objects and the inverse of  $F$  on hom-sets will then be an inverse functor for  $F$ . We can see that  $F$  is injective by a simple application of functional extensionality. If we take  $g : X \times \Theta \rightarrow [0, 1]$  to be a Markov kernel and define  $f(x)(T) = g(x, T)$ , then  $F(f) = g$ , and therefore  $f$  is a measurable map  $X \rightarrow \mathcal{G}(Y)$  by the only if direction of the earlier statements used to prove  $F$  is a functor. Therefore  $F$  is a bijection on hom-sets.  $\square$

The Giry monad originates in [46], which came after [116], where the Radon monad is defined. The equivalence between Markov kernels and Kleisli morphisms of the Giry monad motivated the definition of the Giry monad. In [78], Lawvere defines  $\mathcal{G}(X)$  as a functor, with the correct  $\sigma$ -algebra, in order to produce a right adjoint to the inclusion  $\mathcal{Mes} \rightarrow \mathbf{Markov}$ . Under the isomorphism in Proposition 1.6.3, this adjunction is equivalent to the usual adjunction for a Kleisli category, with functors  $\mathcal{Mes} \rightarrow \mathcal{Kl}(\mathcal{G})$  and a right adjoint  $\mathcal{Kl}(\mathcal{G}) \rightarrow \mathcal{Mes}$ . Lawvere's paper [78] was written before Kleisli's paper on the Kleisli category was published [70].

Using a weak map of monads<sup>3</sup>  $\mathcal{R} \rightarrow \mathcal{G}$ , we can define functors  $\mathcal{EM}(\mathcal{R}) \rightarrow \mathcal{EM}(\mathcal{G})$  and  $\mathcal{Kl}(\mathcal{R}) \rightarrow \mathcal{Kl}(\mathcal{G})$ . We will then relate Theorem 1.5.1 to previous work by Umegaki [119, Theorem 7.1].

We must first discuss the  $\sigma$ -algebra we shall be using on compact Hausdorff spaces, the  $\sigma$ -algebra of *Baire sets*. We require a few definitions from general topology first. In a topological space  $X$ , the *zero sets* are the subsets  $Z \subseteq X$  such that there exists a continuous function  $a : X \rightarrow \mathbb{R}$  such that  $Z = f^{-1}(0)$ . Every zero set is a closed set, and in metrizable spaces every closed set is a zero set [36, Corollary 4.1.12]. As  $\mathbb{R}$  is metrizable, zero sets in an arbitrary space  $X$  can be equivalently characterized as sets of the form  $f^{-1}(C)$  for  $C$  a closed subset of  $\mathbb{R}$ . A set is called  $G_\delta$  if it is expressible as a countable intersection of open sets. In a compact Hausdorff space, the compact  $G_\delta$  subsets are the same as the closed  $G_\delta$  subsets, which are the same as the zero sets [36, Theorem 3.1.10 and Corollary 1.5.12].

Let  $X$  be a compact Hausdorff space. The *Baire  $\sigma$ -algebra*  $\mathcal{Ba}(X)$  is variously defined to be the  $\sigma$ -algebra generated by the zero sets [41, Definition 4A3K (a)], the  $\sigma$ -algebra generated by the compact  $G_\delta$  sets [106, §14.1], or the  $\sigma$ -ring generated by the compact  $G_\delta$  sets [49, §51]. By the previous paragraph, these definitions all coincide for a compact Hausdorff space. Furthermore, if

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<sup>3</sup>see p. 36

$X$  is metrizable, the Baire sets and the Borel sets coincide. The Baire  $\sigma$ -algebra can also be characterized as the coarsest such that every continuous real-valued function is measurable (with respect to the Borel  $\sigma$ -algebra on  $\mathbb{R}$ ) [41, Lemma 4A3L]. We can use the Baire  $\sigma$ -algebra to define a functor  $\text{Ba} : \mathbf{CHaus} \rightarrow \mathcal{Mes}$ :

$$\begin{aligned}\text{Ba}(X) &= (X, \mathcal{Ba}(X)) \\ \text{Ba}(f : X \rightarrow Y) &= f\end{aligned}$$

**Proposition 1.6.4.** *Ba is a functor.*

*Proof.* We have that  $\text{Ba}(X)$  is a measurable space for all  $X \in \mathbf{CHaus}$ , so we only need to focus on the morphisms. First we need to show that  $\text{Ba}(f)$  is measurable for a continuous function  $f : X \rightarrow Y$ . It suffices to show that the preimage of a zero set in  $Y$  is a Baire subset of  $X$ . Let  $T \subseteq Y$  be a zero set defined by the continuous function  $b : Y \rightarrow \mathbb{R}$ , i.e.  $b^{-1}(0) = T$ . Then

$$f^{-1}(T) = f^{-1}(b^{-1}(0)) = (b \circ f)^{-1}(0),$$

which is a zero set in  $X$  because  $b \circ f$  is continuous.

We also have that  $\text{Ba}$  preserves identities and composition because these are defined in the same way in  $\mathbf{CHaus}$  and in  $\mathcal{Mes}$ .  $\square$

Now that we have a functor  $\mathbf{CHaus} \rightarrow \mathcal{Mes}$ , we are ready to define a weak map of monads. For each  $X \in \mathbf{CHaus}$ , we define  $\rho_X : \mathcal{GBa}(X) \rightarrow \text{Ba}(\mathcal{R}(X))$  as

$$\rho_X(\nu)(a) = \int_X a \, d\nu, \tag{1.12}$$

where  $\nu \in \mathcal{G}(\text{Ba}(X))$  and  $a \in C(X)$ .

One version of the Riesz representation theorem uses Baire measures instead of inner regular Borel measures [106, §14.3 Theorem 8]. The statement is exactly that (1.12) is a bijection. In order to prove that this is a weak map of monads, we will require some lemmas.

First, we give a name to a map we have used implicitly in the definition of  $\mu$  for  $\mathcal{R}$ . For a compact Hausdorff space  $X$ , we define  $\zeta_X : C(X) \rightarrow C(\mathcal{R}(X))$  as follows, where  $a \in C(X)$  and  $\phi \in \mathcal{R}(X)$ :

$$\zeta_X(a)(\phi) = \phi(a).$$

The reason that  $\zeta_X(a)$  is always an element of  $C(\mathcal{R}(X))$  is that the weak-\* topology is defined to make  $\zeta_X(a)$  continuous.

**Lemma 1.6.5.** *Let  $X$  be a compact Hausdorff space and  $a$  a continuous function. We have*

$$p_a = \zeta_X(a) \circ \rho_X.$$

*Proof.* Let  $\nu \in \mathcal{G}(\text{Ba}(X))$ , and expand the definitions:

$$\zeta_X(a)(\rho_X(\nu)) = \rho_X(\nu)(a) = \int_X a \, d\nu = p_a(\nu).$$

□

**Lemma 1.6.6.** *Let  $X$  be a topological space, and  $Z \subseteq X$  a zero set. There is a sequence of continuous functions  $(a_i)_{i \in \mathbb{N}} : X \rightarrow [0, 1]$  converging pointwise to  $\chi_Z$ .*

*Proof.* As  $Z$  is a zero set, there exists a continuous function  $a : X \rightarrow \mathbb{R}$  such that  $a^{-1}(0) = Z$ . Using the lattice operations on continuous functions, we can redefine  $a$  to be  $X \rightarrow [0, 1]$  by taking it to be  $|a| \wedge 1$ . This does not change where  $a$  is zero, so  $a^{-1}(0) = Z$  is preserved. We can then define

$$a_i = (1 - a)^i$$

for all  $i \in \mathbb{N}$ . This is a family of continuous functions, taking the value 1 on  $Z$  and some value in  $[0, 1)$  outside of  $Z$ . For all  $\alpha \in [0, 1)$ , we have  $\alpha^n \rightarrow 0$  as  $n \rightarrow \infty$ , so  $a_i \rightarrow \chi_Z$  pointwise. □

**Lemma 1.6.7.** *For each compact Hausdorff space  $X$ , the ring generated by the elements  $\{\zeta_X(a)\}_{a \in C(X)}$  is dense in  $C(\mathcal{R}(X))$ .*

*Proof.* By the Stone-Weierstrass theorem [20, Theorem V.8.1], it suffices to show that elements of the form  $\zeta_X(a)$  for  $a \in C(X)$  separate the points of  $\mathcal{R}(X)$ . As elements of  $\mathcal{R}(X)$  are defined to be functions  $C(X) \rightarrow \mathbb{R}$ , this follows from functional extensionality. □

**Theorem 1.6.8.** *As defined above,  $(\text{Ba}, \rho)$  is a weak map of monads  $\mathcal{R} \rightarrow \mathcal{G}$ .*

*Proof.* As already discussed, (1.12) is a bijection for each  $X \in \mathbf{CHaus}$  (a version of the Riesz representation theorem). We need to show that it is an isomorphism in  $\mathcal{Mes}$ , i.e. that it is measurable and its inverse is measurable.

We first show that  $\rho_X^{-1}$  is measurable. It suffices to show this on a set of generators for the  $\sigma$ -algebra on  $\mathcal{G}(\text{Ba}(X))$ , i.e. that for each  $S \in \mathcal{Ba}(X)$  and  $B \in \mathcal{Bo}([0, 1])$  we have  $\rho_X(p_S^{-1}(B)) \in \mathcal{Ba}(\mathcal{R}(X))$ .

By Lemma 1.6.5, for each continuous function  $a \in C(X)$ , we have  $p_a = \zeta_X(a) \circ \rho_X$ , and therefore  $p_a \circ \rho_X^{-1} = \zeta_X(a)$ . As  $\zeta_X(a) \in C(\mathcal{R}(X))$ , we have

that  $p_a \circ \rho_X^{-1}$  is also, and is therefore an element of  $\mathcal{L}^\infty(\text{Ba}(\mathcal{R}(X)))$ . For each zero set  $Z \subseteq X$ , there is a sequence of continuous functions  $a_i \rightarrow \chi_Z$  (Lemma 1.6.6), so  $p_{a_i} \rightarrow p_{\chi_Z}$  by Lemma 1.6.1, so  $p_{a_i} \circ \rho_X^{-1} \rightarrow p_Z \circ \rho_X^{-1}$  pointwise, and therefore  $p_Z \circ \rho_X^{-1}$  is  $\mathcal{Ba}(\mathcal{R}(X))$ -measurable.

Since for any sequence of sets  $(S_i)_{i \in \mathbb{N}}$ , where  $S_i \in \mathcal{Ba}(X)$ , we have  $\chi_{\bigcup_{i=1}^n S_i}$  converges pointwise to  $\chi_{\bigcup_{i=1}^\infty S_i}$ , we can use Lemma 1.6.1 to conclude that  $p_{\bigcup_{i=1}^n S_i} \circ \rho_X^{-1} \rightarrow p_{\bigcup_{i=1}^\infty S_i} \circ \rho_X^{-1}$ . We can also use the linearity from Lemma 1.6.1 to show that if  $p_S \circ \rho_X^{-1}$  is measurable, so is  $p_{-S} \circ \rho_X^{-1}$ . As  $\mathcal{Ba}(X)$  is built up from zero sets via countable unions and complements, we have shown that for any  $S \in \mathcal{Ba}(X)$ , we have that  $p_S \circ \rho_X^{-1} : \text{Ba}(\mathcal{R}(X)) \rightarrow [0, 1]$  is measurable (with the Borel  $\sigma$ -algebra on  $[0, 1]$ ). Therefore, for each  $B \in \mathcal{Bo}([0, 1])$

$$\rho_X(p_S^{-1}(B)) = (\rho_X^{-1})^{-1}(p_S^{-1}(B)) = (p_S \circ \rho_X^{-1})^{-1}(B) \in \mathcal{Ba}(\mathcal{R}(X)),$$

as is required to prove that  $\rho_X^{-1}$  is measurable.

To show that  $\rho_X$  is measurable, it suffices to show that the preimage of a zero set in  $\mathcal{R}(X)$  is measurable in  $\mathcal{G}(\text{Ba}(X))$ . In turn, it suffices to show that for each continuous function  $b : \mathcal{R}(X) \rightarrow \mathbb{R}$ , the function  $b \circ \rho_X$  is  $\mathcal{G}(\text{Ba}(X))$ -measurable. This is what we prove.

To start with, we have that if  $b = \zeta_X(a)$  for some  $a \in C(X)$ , then  $b \circ \rho_X$  is  $\mathcal{G}(\text{Ba}(X))$ -measurable because it is equal to  $p_a$  by Lemma 1.6.5. Now, measurable real-valued functions are closed under pointwise ring operations, because the ring operations on  $\mathbb{R}$  are measurable. Therefore, if  $b$  is in the ring generated by  $\{\zeta_X(a)\}_{a \in C(X)}$ , then  $b \circ \rho_X$  is  $\mathcal{G}(\text{Ba}(X))$ -measurable. Finally, by Lemma 1.6.7, for each  $b \in C(\mathcal{R}(X))$ , there is a sequence  $(b_i)_{i \in \mathbb{N}}$ , where each  $b_i$  is in the ring generated by  $\{\zeta_X(a)\}_{a \in C(X)}$ , such that  $b_i \rightarrow b$  uniformly, and therefore pointwise. Therefore  $b \circ \rho_X$  is  $\mathcal{G}(\text{Ba}(X))$ -measurable for all  $b \in C(\mathcal{R}(X))$ . This concludes the proof that  $\rho_X$  is a measurable isomorphism for all  $X \in \mathbf{CHaus}$ .

To show that  $(\rho_X)$  defines a natural transformation, we need to show that for each continuous map of compact Hausdorff spaces  $f : X \rightarrow Y$ , the diagram

$$\begin{array}{ccc} \mathcal{G}(\text{Ba}(X)) & \xrightarrow{\rho_X} & \text{Ba}(\mathcal{R}(X)) \\ \mathcal{G}(\text{Ba}(f)) \downarrow & & \downarrow \text{Ba}(\mathcal{R}(f)) \\ \mathcal{G}(\text{Ba}(Y)) & \xrightarrow{\rho_Y} & \text{Ba}(\mathcal{R}(Y)) \end{array}$$

commutes. Let  $\nu \in \mathcal{G}(\text{Ba}(X))$  and  $b \in C(Y)$ , and starting with the lower left

route

$$\begin{aligned}
 \rho_Y(\mathcal{G}(\text{Ba}(f))(\nu))(b) &= \rho_Y(\mathcal{G}(f)(\nu))(b) \\
 &= \int_Y b \, d\mathcal{G}(f)(\nu) \\
 &= \int_X b \circ f \, d\nu && \text{by Proposition 1.6.2 (i)} \\
 &= \rho_X(\nu)(b \circ f) \\
 &= \mathcal{R}(f)(\rho_X(\nu))(b) \\
 &= \text{Ba}(\mathcal{R}(f))(\rho_X(\nu))(b),
 \end{aligned}$$

which shows that the diagram commutes.

The diagram we need for units (see (0.4)) is

$$\begin{array}{ccc}
 \text{Ba}(X) & \xrightarrow{\eta_{\text{Ba}(X)}^{\mathcal{G}}} & \mathcal{G}(\text{Ba}(X)) \\
 & \searrow \text{Ba}(\eta_X^{\mathcal{R}}) & \downarrow \sigma_X \\
 & & \text{Ba}(\mathcal{R}(X)).
 \end{array}$$

To show that this commutes, let  $x \in X$  and  $a \in C(X)$ , and observe that

$$\begin{aligned}
 \sigma_X(\eta_{\text{Ba}(X)}^{\mathcal{G}}(x))(a) &= \int_X a \, d\eta_{\text{Ba}(X)}^{\mathcal{G}}(x) \\
 &= a(x) && \text{by Proposition 1.6.2 (ii)} \\
 &= \eta_X^{\mathcal{R}}(x)(a),
 \end{aligned}$$

showing that the diagram commutes.

The diagram we need for the multiplications (see (0.4)) is

$$\begin{array}{ccccc}
 \mathcal{G}^2(\text{Ba}(X)) & \xrightarrow{\mathcal{G}(\sigma_X)} & \mathcal{G}(\text{Ba}(\mathcal{R}(X))) & \xrightarrow{\sigma_{\mathcal{R}(X)}} & \text{Ba}(\mathcal{R}^2(X)) \\
 \mu_{\text{Ba}(X)}^{\mathcal{G}} \downarrow & & & & \downarrow \text{Ba}(\mu_X^{\mathcal{R}}) \\
 \mathcal{G}(\text{Ba}(X)) & \xrightarrow{\sigma_X} & & & \text{Ba}(\mathcal{R}(X)).
 \end{array}$$

To show that this commutes, let  $\Phi \in \mathcal{G}^2(\text{Ba}(X))$  and  $a \in C(X)$ , and start

with the bottom left path

$$\begin{aligned}
\sigma_X(\mu_{\text{Ba}(X)}^{\mathcal{G}}(\Phi))(a) &= \int_X a \, d\mu_{\text{Ba}(X)}^{\mathcal{G}}(\Phi) \\
&= \int_{\mathcal{G}(\text{Ba}(X))} p_a \, d\Phi && \text{Proposition 1.6.2 (iii)} \\
&= \int_{\mathcal{G}(\text{Ba}(X))} \zeta_X(a) \circ \rho_X \, d\Phi && \text{Lemma 1.6.5} \\
&= \int_{\mathcal{R}(X)} \zeta_X(a) \, d\mathcal{G}(\sigma_X)(\Phi) && \text{Proposition 1.6.2 (i)} \\
&= \sigma_{\mathcal{R}(X)}(\mathcal{G}(\sigma_X)(\Phi))(\zeta_X(a)) \\
&= \text{Ba}(\mu_X^{\mathcal{R}})(\sigma_{\mathcal{R}(X)}(\mathcal{G}(\sigma_X)(\Phi)))(a),
\end{aligned}$$

which shows that the diagram commutes. Therefore  $\sigma$  is a natural isomorphism and a lax map of monads, and therefore a weak map of monads.  $\square$

By Proposition 0.4.7, we therefore have functors  $\text{Ba}^\rho : \mathcal{EM}(\mathcal{R}) \rightarrow \mathcal{EM}(\mathcal{G})$  and  $\text{Ba}_{\rho^{-1}} : \mathcal{Kl}(\mathcal{R}) \rightarrow \mathcal{Kl}(\mathcal{G})$ . We can also form the composite  $F \circ \text{Ba}_{\rho^{-1}} : \mathcal{Kl}(\mathcal{R}) \rightarrow \mathbf{Markov}$ . The image of this functor consists of Markov kernels of the form  $X \times \mathcal{Ba}(Y) \rightarrow [0, 1]$ . Umegaki [119] calls these channels and proves that for each  $f : X \times \mathcal{Ba}(Y) \rightarrow [0, 1]$ , we can define a positive unital  $\sigma$ -normal linear map  $K : \mathcal{L}^\infty(Y, \mathcal{Ba}(Y)) \rightarrow \mathcal{L}^\infty(X, \mathcal{Ba}(X))$ , as

$$K(b)(x) = \int_Y b \, df(x)(-),$$

where a map  $g : \mathcal{L}^\infty(Y, \mathcal{Ba}(Y)) \rightarrow \mathcal{L}^\infty(X, \mathcal{Ba}(X))$  is said to be  $\sigma$ -normal if for every decreasing sequence  $(b_i)$  in  $\mathcal{L}^\infty(Y, \mathcal{Ba}(Y))$  with infimum 0, the sequence  $g(b_i)$  has infimum 0 in  $\mathcal{L}^\infty(X, \mathcal{Ba}(X))$ . It is easy to see, using linearity, that this is equivalent to the preservation of bounded suprema and infima of monotone sequences.

Umegaki then shows that this correspondence is an isomorphism [119, Theorem 7.1]. This is a forerunner of later results [75, §2][92, Theorem 5.16] working with arbitrary measurable spaces. Note, however, that Umegaki does not consider composition of Markov kernels (or the Chapman-Kolmogorov equation) at any point in his article, so Umegaki stops short of describing an equivalence of categories.

If, taking a morphism  $f : X \rightarrow \mathcal{R}(Y)$  in  $\mathcal{Kl}(\mathcal{R})$ , we apply Umegaki's correspondence to  $F(\text{Ba}_{\rho^{-1}}(f))$  we get a positive unital map  $\mathcal{L}^\infty(Y) \rightarrow \mathcal{L}^\infty(X)$ . If we had applied Theorem 1.5.1, we would have got a map  $C(Y) \rightarrow C(X)$ . By using *continuous Markov kernels*, as suggested by the thesis committee, we can adapt Umegaki's result into an equivalence. We define the category of continuous Markov kernels **CMarkov** to have compact Hausdorff spaces as objects, and a map from  $X \rightarrow Y$  to be a Markov kernel  $f : X \times \mathcal{B}a(Y) \rightarrow [0, 1]$  such that for all  $b \in C(Y)$ , the map

$$x \rightarrow \int_Y b \, df(x, -)$$

is continuous<sup>4</sup>. We define a functor  $G : \mathcal{Kl}(\mathcal{R}) \rightarrow \mathbf{CMarkov}$ , as the identity on objects and  $F \circ \text{Ba}_{\rho^{-1}}$  on morphisms.

We require a lemma.

**Lemma 1.6.9.** *Let  $X$  be a compact Hausdorff space,  $\phi \in \mathcal{R}(X)$ , and  $a \in C(X)$ . Then*

$$\int_X a \, d\rho_X^{-1}(\phi) = \phi(a)$$

*Proof.* As  $\phi = \rho_X(\rho_X^{-1}(\phi))$ , we have

$$\phi(a) = \rho_X(\rho_X^{-1}(\phi)) = \int_X a \, d\rho_X^{-1}(\phi).$$

□

**Proposition 1.6.10.** ***CMarkov** is a category, and  $G$  an equivalence of categories.*

*Proof.* We take identities in **CMarkov** to agree with (1.10). We then have that for all compact Hausdorff  $X$ ,  $a \in C(X)$  and  $x \in X$

$$\int_X a \, d(\text{id}_X(x, -)) = a(x),$$

by Proposition 1.6.2 (ii), so the identities are continuous Markov kernels. The above also shows that  $G$  preserves identity morphisms. We define composition

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<sup>4</sup>This usually occurs in the stochastic process literature as part of the definition of a *Feller process* [18, §2.2 Definition (i)] [30, §2.7] [38] [39, §X.8].



in **CMarkov** by (1.9), as in **Markov**. The way we prove that the composition of two continuous Markov kernels is continuous is by proving the statements necessary to show that  $G$  is an equivalence, which imply this and the rest of the statements necessary to show **CMarkov** is a category.

If  $f : X \rightarrow \mathcal{R}(Y)$  is a map in  $\mathcal{Kl}(\mathcal{R})$ , we need to show that  $G(f) = F(\text{Ba}_{\rho^{-1}}(f))$  is continuous (as a Markov kernel). If we let  $b \in C(Y)$ , we observe that

$$\int_Y b \, dF(\text{Ba}_{\rho^{-1}}(f))(x, -) = \int_Y b \, d\rho_Y^{-1}(f(x)) = f(x)(b),$$

by Lemma 1.6.9. We already saw in the proof of Theorem 1.5.1 that  $f(-)(b) : X \rightarrow \mathbb{C}$  is continuous, so  $G(f)$  is a continuous Markov kernel.

We prove the injectivity of  $G$  on morphisms as follows. Let  $f, g : X \rightarrow \mathcal{R}(Y)$ , and suppose that  $G(f) = G(g)$ . Then for all  $x \in X$  and  $b \in C(Y)$ , we have

$$f(x)(b) = \int_Y b \, dG(f)(x, -) = \int_Y b \, dG(g)(x, -) = g(x)(b),$$

so  $f = g$ .

To prove the surjectivity of  $G$  on morphisms, let  $g : X \times \mathcal{B}a(Y) \rightarrow [0, 1]$  be a continuous Markov kernel. Define  $f : X \rightarrow \mathcal{R}(Y)$  for all  $x \in X$  and  $b \in C(Y)$  as

$$f(x)(b) = \int_Y b \, dg(x, -).$$

We have that  $f(x) \in \mathcal{R}(Y)$  by linearity of integration and the fact that  $g(x, -)$  is a probability measure. If  $x_i \rightarrow x$  in  $X$ , then for all  $b \in C(Y)$  we have

$$\int_Y b \, dg(x_i, -) \rightarrow \int_Y b \, dg(x, -)$$

by the definition of a continuous Markov kernel. Therefore  $f$  is a continuous map from  $X \rightarrow \mathcal{R}(Y)$  with respect to the weak-\* topology, and so  $f$  is a Kleisli map in  $\mathcal{Kl}(\mathcal{R})$ . Now, for all  $x \in X$  and  $b \in C(Y)$ , we have

$$\int_Y b \, dG(f)(x, -) = f(x)(b) = \int_Y b \, dg(x, -).$$

By the Baire measure form of the Riesz representation theorem, we therefore have  $G(f)(x, -) = g(x, -)$ , and as this holds for all  $x \in X$ ,  $G(f) = g$ , as required.

As  $G$  was defined on morphisms as the composite of two functors, it preserves identities and composition. When combined with what we have proven above, this shows that **CMarkov** is a category and  $G$  is an equivalence (in fact, an isomorphism).  $\square$

We can then define a functor  $L : \mathbf{CMarkov} \rightarrow \mathbf{CC}^*\mathbf{Alg}_{PU}^{\text{op}}$  as follows. On objects,  $L(X) = C(X)$ . On morphisms, if  $f : X \times \mathcal{B}a(Y) \rightarrow [0, 1]$  is a continuous Markov kernel,

$$L(f)(b)(x) = \int_Y b \, df(x, -),$$

where  $b \in C(X)$  and  $x \in X$ , like Umegaki's definition in [119, (7.1)].

**Proposition 1.6.11.**  *$L$  is a functor.*

*Proof.* If  $f : X \times \mathcal{B}a(Y) \rightarrow [0, 1]$  is a continuous Markov kernel, then  $L(f)$  is a linear positive unital map  $C(Y) \rightarrow C(X)$  by the linearity of integration and the fact that  $f(x, -)$  is a probability measure. We have that  $L$  preserves identity maps by Proposition 1.6.2 (ii). To prove that  $L$  preserves composition, it suffices to show that for continuous Markov kernels  $f : X \times \mathcal{B}a(Y) \rightarrow [0, 1]$  and  $g : Y \times \mathcal{B}a(Z) \rightarrow [0, 1]$ , for all  $c \in C(Z)$  and  $x \in X$  we have

$$\int_Z c(z)(g * f)(x, dz) = \int_Y \left( \int_Z c(z)g(y, dz) \right) f(x, dy),$$

where we use dummy variables  $y \in Y$  and  $z \in Z$  for convenience. We do this by proving it for all  $c \in \mathcal{L}^\infty(Z, \mathcal{B}a(Z))$  using a standard argument with the dominated convergence theorem, as in the proofs in Proposition 1.6.2. We show it for  $\chi_U$ , where  $U \in \mathcal{B}a(Z)$  as follows:

$$\begin{aligned} \int_Z \chi_U(g * f)(x, dz) &= (g * f)(x, U) = \int_Y g(y, U) f(x, dy) \\ &= \int_Y \left( \int_Z \chi_U g(y, dz) \right) f(x, dy), \end{aligned}$$

and it then follows for simple functions by linearity, and then for all measurable functions by the dominated convergence theorem.  $\square$

We can now prove that  $L$  is an equivalence of categories.

**Proposition 1.6.12.**  *$L \circ G = \mathcal{C}_{\mathcal{R}}$ . Therefore  $L$  is an equivalence.*

*Proof.* On objects, we have  $L(G(X)) = L(X) = C(X) = \mathcal{C}_{\mathcal{R}}(X)$  for all compact Hausdorff  $X$ . For a Kleisli map  $f : X \rightarrow \mathcal{R}(Y)$ , we have, for all  $b \in C(Y)$

and  $x \in X$ :

$$\begin{aligned}
 L(G(f))(b)(x) &= \int_Y b \, dG(x, -) \\
 &= \int_Y b \, d\rho_X^{-1}(f(x)) \\
 &= f(x)(b) && \text{Lemma 1.6.9} \\
 &= \mathcal{C}_{\mathcal{R}}(f)(b)(x).
 \end{aligned}$$

As  $G$  and  $\mathcal{C}_{\mathcal{R}}$  are categorical equivalences (Proposition 1.6.10 and Theorem 1.5.1),  $L$  is also an equivalence.  $\square$

Alternatively we could prove that  $L$  is an equivalence using Umegaki's [119, Theorem 7.1] together with the fact that any linear positive unital map  $C(Y) \rightarrow C(X)$  extends to a map  $\mathcal{L}^\infty(Y, \mathcal{B}a(Y)) \rightarrow \mathcal{L}^\infty(X, \mathcal{B}a(X))$  that is linear, positive, unital and preserves infima of bounded monotone sequences converging to 0, which follows from [91, Lemma 3]. Then Proposition 1.6.12 provides an alternative proof of Theorem (1.5.1) from Umegaki's result.



## Chapter 2

# Base-Norm Spaces

### 2.1 Introduction

In this chapter, we consider the notion of a *base-norm space* and its relationship to convex sets, the distribution monad, and order-unit spaces. Just as order-unit spaces are non-multiplicative, order-theoretic generalizations of the notion of  $C^*$ -algebra and  $W^*$ -algebra (Proposition 1.2.10 is a justification of this view), base-norm spaces are similar generalizations of the dual of a  $C^*$ -algebra, the Banach space containing the state space, and the predual of a  $W^*$ -algebra, which likewise contains the normal state space (up to isomorphism).

In the literature, there are several definitions of base-norm space, falling into three equivalence classes. Only one of these equivalence classes of definitions is suitable for duality with order-unit spaces, as we shall see, and this is what we choose to call a base-norm space (forming a category **BNS**). The kind of space corresponding to the least strict notion of “base norm space” used in the literature is what we call a pre-base-norm space (forming a category **PreBNS**).

We also show that we can embed any bounded convex set into a pre-base-norm space, in a different way from [47, Theorem 2.2], and we show that this embedding forms an equivalence of categories. We then prove that any sequentially complete bounded convex set embeds as the base of a Banach base-norm space, slightly generalizing Gudder’s [47, Theorem 3.6]. This is the relationship between the base-norm space approach to generalized probabilistic theories from Davies and Lewis [23] and Edwards [31] and convex-set-based

approach [12, 11], used more recently. This is also described in [47, 91, 123, 9]. Unfortunately, for reasons of space, we cannot treat tensor products of base-norm spaces and order-unit spaces. If we did, it would go along the lines of [89] and [123].

We construct the left adjoints to the functors  $B : \mathbf{PreBNS} \rightarrow \mathbf{Set}$  and  $B : \mathbf{BBNS} \rightarrow \mathbf{Set}$  and show that the monads arising from these adjunctions are the familiar discrete distribution monads, and that the comparison functors to  $\mathcal{EM}(\mathcal{D})$  and  $\mathcal{EM}(\mathcal{D}_\infty)$  respectively are full and faithful. We then construct the functor  $\mathbf{BAff}$ , taking bounded affine functions on a  $\mathcal{D}$ -algebra, and show that when this is applied to the base of a pre-base-norm space it gives the dual space. The definition of the order-unit space  $\mathbf{BAff}$  in the case of abstract convex was first given in [91, Theorem 2] in the setting of convex prestructures, which are a notion of convex set including all  $\mathcal{D}$ -algebras. The special case of  $\mathbf{BAff}(X)$  for  $X$  the base of a base-norm space occurred earlier [31, p. 211].

Finally, we show that taking dual spaces defines functors  $F : \mathbf{PreBNS} \rightarrow \mathbf{OUS}^{\text{op}}$  and  $G : \mathbf{OUS}^{\text{op}} \rightarrow \mathbf{PreBNS}$ , and  $F$  is a left adjoint to  $G$ , in a variant of the adjunctions defined in [52, Theorem 17] and [59, Proposition 5]. We then briefly discuss how this can be restricted to an equivalence. In later chapters this adjunction will be generalized in two different ways to give two equivalences of categories.

## 2.2 Definitions

*As this section is introductory, it does not contain any original results.*

### 2.2.1 Base-Norm Spaces

A *pre-base-norm* space consists of a triple  $(E, E_+, \tau)$ , where  $(E, E_+)$  is a directed ordered vector space, and  $\tau : E \rightarrow \mathbb{R}$  is a positive linear functional that is not the zero linear functional unless  $E = \{0\}$ . The map  $\tau$  is called the trace and is subject to another axiom that we describe below. The *base* is

$$B = \tau^{-1}(1) \cap E_+.$$

The reason for this definition can be seen by considering  $C^*$ -algebras. If  $E = A^*$ , for  $A$  a (unital)  $C^*$ -algebra<sup>1</sup>, and  $\tau(\phi) = \phi(1)$  for  $\phi \in A^*$ ,  $B$  is the state space of  $A$ . In the commutative case the base is the set of Radon probability measures sitting inside the vector space of signed Radon measures.

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<sup>1</sup>Recall that  $A^*$  means the vector space of continuous linear functionals (Section 0.3), where the topology on  $A$  is that determined by the norm.

Given the base  $B$ , we define the *unit ball*  $U$  to be the absolutely convex hull of  $B$ . In the case that  $B$  is non-empty, by Lemma 0.1.1 this can equivalently be defined as

$$U = \text{co}(B \cup -B).$$

For  $E$  to be a *pre-base-norm space* we require  $U$  to be radially bounded, *i.e.* each ray in  $E$  intersects  $U$  in a bounded subset (considering the ray as isomorphic to  $\mathbb{R}$ ).

In summary, a pre-base-norm space is a triple  $(E, E_+, \tau)$  such that  $(E, E_+)$  is a directed ordered vector space,  $\tau$  is a positive linear functional, non-zero if  $E \neq \{0\}$ , and  $U$  is radially bounded.

To live up to their name, pre-base-norm spaces should have an intrinsic notion of norm. We therefore want to show that  $U$  is absorbent so that we can define the norm as its Minkowski functional, which will then be a norm by Lemma 0.1.2. We must take a slight detour first.

**Lemma 2.2.1.** *For any pre-base-norm space, if  $B$  is empty,  $E = \{0\}$ .*

*Proof.* Suppose for a contradiction that  $B$  is empty but  $E \neq \{0\}$ . Since  $\tau$  is a trace, we must have  $\tau \neq 0$ , which means there is  $x \in E$ ,  $x \neq 0$ , such that  $\tau(x) \neq 0$ . Since  $(E, E_+)$  is directed,  $x = x_+ - x_-$  for  $x_+, x_- \in E_+$ , and at least one of  $y = x_{\pm}$  must be non-zero and satisfy  $\tau(y) \neq 0$ . But then  $z = \frac{y}{\tau(y)}$  is in  $B$  because  $z$  is positive and

$$\tau(z) = \tau\left(\frac{y}{\tau(y)}\right) = \frac{\tau(y)}{\tau(y)} = 1,$$

contradicting our initial assumption.  $\square$

We say a positive linear functional is *strictly* positive if  $x \in E_+$ ,  $\tau(x) = 0$  implies  $x = 0$ .

**Lemma 2.2.2.** *The trace  $\tau : E \rightarrow \mathbb{R}$  on any pre-base-norm space is strictly positive.*

*Proof.* If  $E = \{0\}$ , then the only possible  $\tau$  is 0, which is strictly positive because  $x \in E$  is always 0. So we therefore consider the case  $E \neq \{0\}$ . Suppose for a contradiction that  $\tau$  is not strictly positive. Then there is an  $x \in E_+$  such that  $\tau(x) = 0$  but  $x \neq 0$ . By Lemma 2.2.1, there is some  $y \in B$ .

By the linearity of the trace,  $\alpha x + y \in B$  for all  $\alpha \in [0, \infty)$ , and  $\beta x - y \in -B$  for all  $\beta \in (-\infty, 0]$  similarly. Therefore

$$U \ni \frac{1}{2}(\alpha x + y) + \frac{1}{2}(\beta x - y) = \left(\frac{1}{2}\alpha + \frac{1}{2}\beta\right)x.$$

If we have  $\gamma \in \mathbb{R}$ , we can write it as  $(\frac{1}{2}(2\gamma) + \frac{1}{2}0)$  if  $\gamma \geq 0$  or  $(\frac{1}{2}0 + \frac{1}{2}(2\gamma))$  if  $\gamma \leq 0$ , and so  $U$  contains the whole of a non-trivial ray, contradicting radial boundedness of  $U$ .  $\square$

**Lemma 2.2.3.** *The set  $U$  in a pre-base-norm space is absorbent.*

*Proof.* Let  $x \in E$ . We need to find  $\alpha \in [0, \infty)$  such that  $x \in \alpha U$ . Take the decomposition  $x = x_+ - x_-$  for  $x_+, x_- \in E_+$ . Define  $\tau(x_+) = \beta$  and  $\tau(x_-) = \gamma$ . If  $\beta \neq 0$ , we have  $\tau(\frac{x_+}{\beta}) = 1$ , so  $x_+ \in \beta U$  and hence  $x_+ \in \beta U$ . If  $\beta = 0$ , by strict positivity (Lemma 2.2.2)  $x_+ = 0$ , so we can redefine  $\beta = 1$  and hence  $x_+ \in \beta U$  in this case as well. Similarly, we have  $x_- \in \gamma U$ .

Define  $\alpha = 2 \max\{\beta, \gamma\}$ . By absolute convexity of  $U$ , we have  $x_+ \in \frac{\alpha}{2}U$  and  $x_- \in \frac{\alpha}{2}U$ , and hence  $2x_{\pm} \in \alpha U$ . We can then apply absolute convexity of  $U$  again to conclude that

$$x = \frac{1}{2}(2x_+) - \frac{1}{2}(2x_-) \in \alpha U,$$

as required.  $\square$

Thus the Minkowski functional  $\|-\|_U$  is always a norm in a pre-base-norm space. We can now define a *base-norm space* – it is a pre-base-norm space in which the positive cone is  $\|-\|_U$ -closed. We call a base-norm space a *Banach base-norm space* if it is complete in this norm (it is also sometimes simply called a complete base-norm space). The analogous notion for pre-base-norm spaces is a *Banach pre-base-norm space*, though this is less useful.

We can now define morphisms of (pre-)base-norm spaces. If  $(E, E_+, \tau)$  and  $(F, F_+, \sigma)$  are (pre-)base-norm spaces, a morphism  $f : E \rightarrow F$  is a linear, positive map that preserves the trace, *i.e.*  $\tau = \sigma \circ f$ . Using these morphisms we form the category **PreBNS**, its full subcategory on base-norm spaces **BNS**, and its full subcategory on Banach base-norm spaces, **BBNS**. These morphisms are the *trace-preserving* morphisms. We also have *trace-reducing* or *trace-decreasing* morphisms, which are required to be positive and for which  $\sigma(f(x)) \leq \tau(x)$  for all  $x \in E_+$ . The category of pre-base-norm spaces and trace-reducing maps will be called **PreBNS** $_{\leq 1}$ , and **BNS** $_{\leq 1}$  and **BBNS** $_{\leq 1}$  are the corresponding full subcategories.

**Lemma 2.2.4.** *For  $\alpha \geq 0$ , if  $x \in \alpha U$ ,  $|\tau(x)| \leq \alpha$ . Therefore  $\|\tau\|$  in the operator norm is  $\leq 1$ , and so  $\tau$  is norm-continuous.*

*Proof.* We show that if  $x \in U$ ,  $|\tau(x)| \leq 1$  and the statement follows by scaling. The element  $x$  is either 0 or is expressible as an element of  $\text{co}(B \cup -B)$ . In the



first case,  $\tau(x) = 0$  and so  $|\tau(x)| \leq \alpha$ . Therefore we concern ourselves with the second case only from now on.

Since  $x \in \text{co}(B \cup -B)$  and  $B$  is convex, any convex combination used to express  $x$  can be reduced to  $\beta x_+ + (1 - \beta)x_-$  for  $x_+ \in B$  and  $x_- \in -B$ , for some  $0 \leq \beta \leq 1$ . Then we have

$$\begin{aligned}\tau(x) &= \tau(\beta x_+ + (1 - \beta)x_-) = \beta\tau(x_+) + (1 - \beta)\tau(x_-) = \beta + (1 - \beta)(-1) \\ &= 2\beta - 1.\end{aligned}$$

From the constraint on  $\beta$  we deduce

$$0 \leq \beta \leq 1 \Leftrightarrow 0 \leq 2\beta \leq 2 \Leftrightarrow -1 \leq 2\beta - 1 \leq 1 \Leftrightarrow |2\beta - 1| \leq 1,$$

which, combined with the previous statement gives  $|\tau(x)| \leq 1$ . By applying Lemma 0.1.8 with  $u$  and  $[-1, 1]$  as the absolutely convex sets, we conclude that  $\|\tau\| \leq 1$ .  $\square$

**Corollary 2.2.5.** *If  $x \in E_+$ ,  $\|x\| = \tau(x)$ .*

*Proof.* We have that  $x \in \tau(x)B$ , and so  $x \in \tau(x)U$ . This shows that  $\|x\| \leq \tau(x)$ . If it were the case that  $\|x\|_U < \tau(x)$ , then

$$\begin{aligned}\inf\{\lambda > 0 \mid x \in \lambda U\} &\not\geq \tau(x) \Leftrightarrow \neg(\forall \lambda > 0. x \in \lambda U \Rightarrow \tau(x) \leq \lambda) \\ &\Leftrightarrow \exists \lambda > 0. x \in \lambda U \wedge \lambda < \tau(x).\end{aligned}$$

Lemma 2.2.4 shows that  $x \in \lambda U$  implies  $\tau(x) \leq \lambda$ , a contradiction, so we have  $\|x\|_U \geq \tau(x)$ , and so  $\|x\| = \tau(x)$ .  $\square$

We can show the following in the case of a pre-base-norm space with a radially compact ball. The first statement below is an elaboration of a standard fact about radially compact pre-base-norm spaces ([4, Proposition II.1.14] or [6, Proposition 1.26]), but we give the proof here for ease of reference. The second statement can be proved as a consequence of duality results between base-norm and order-unit spaces as defined by Alfsen and Shultz [6, Corollary 1.27], but the proof below is elementary.

**Proposition 2.2.6.** *Let  $(E, E_+, \tau)$  be a pre-base-norm space such that the base  $B$  is nonempty and  $U = \text{absco}(B)$  is radially compact.*

- (i) *Every  $x \in E$  can be expressed as  $\alpha x_+ - (1 - \alpha)x_-$  where  $\alpha \in [0, 1]$ ,  $x_+, x_- \in E_+$  and  $\|x_+\| = \|x_-\| = \|x\|$ . The  $\alpha$  is uniquely determined if  $x \neq 0$ , and is equal to  $\frac{1}{2}(\frac{\tau(x)}{\|x\|} + 1)$ .*

(ii)  $E_+$  is closed, and therefore  $(E, E_+, \tau)$  is a base-norm space.

*Proof.*

- (i) As  $U$  is radially compact, we have that  $x \in \|x\|U$  (Lemma 0.1.7). Since  $B$  is nonempty,  $U = \text{co}(-B \cup B)$  (Lemma 0.1.1). Because  $B$  is convex, we can therefore express  $x = \alpha x_+ - (1 - \alpha)x_-$  with  $\alpha \in [0, 1]$  and  $x_+, x_- \in \|x\|B$ . By Corollary 2.2.5,  $\|x_\pm\| = \tau(x_\pm) = \|x\|$ .

Note that we have

$$\begin{aligned} \tau(\alpha x_+ - (1 - \alpha)x_-) &= \alpha\tau(x_+) - (1 - \alpha)\tau(x_-) \\ &= \alpha\|x_+\| - (1 - \alpha)\|x_-\| \\ &= (2\alpha - 1)\|x\|. \end{aligned}$$

If  $x \neq 0$ , we have  $\|x\| \neq 0$  so we can rearrange this expression to get  $\alpha = \frac{1}{2}(\frac{\tau(x)}{\|x\|} + 1)$ . The expression on the right depends only on  $x$ , so  $\alpha$  is uniquely defined.

- (ii) Let  $(x_i)$  be a sequence in  $E_+$  converging in the base-norm to  $x \in E$ . If  $x = 0$  then  $x \in E_+$ , so we reduce to the case that  $x \neq 0$ . Observe that if this is so  $\alpha = 1$  iff  $x \in E_+$ , because if  $\alpha = 1$  then we have  $x = x_+ \in E_+$ , and if  $x \in E_+$  it is expressible as  $1 \cdot x - 0 \cdot x$ , which by uniqueness of  $\alpha$  gives  $\alpha = 1$ . We also observe that  $\|\cdot\|$  is continuous on  $E$ , so  $\|\cdot\|^{-1}$  is continuous on  $E \setminus \{0\}$ . As  $\tau$  is continuous (Lemma 2.2.4), we have that  $x \mapsto \frac{1}{2}(\frac{\tau(x)}{\|x\|} + 1)$  is continuous on  $E \setminus \{0\}$ . As  $x \neq 0$ , we can replace  $(x_i)$  with a subsequence  $(y_i)$  such that  $\|x - x_i\| \leq \frac{\|x\|}{2}$  and therefore  $y_i \neq 0$  for all  $i$ . Then we have  $\frac{1}{2}(\frac{\tau(y_i)}{\|y_i\|} + 1) = 1$  for all  $y_i$  and so we must have  $\frac{1}{2}(\frac{\tau(x)}{\|x\|} + 1) = 1$  for  $x$  by continuity, implying  $x \in E_+$ .  $\square$

We now move on to proving facts about morphisms.

**Lemma 2.2.7.** *A trace-preserving morphism  $f : (E, E_+, \tau) \rightarrow (F, F_+, \sigma)$  of pre-base-norm spaces maps the base into the base, i.e. if  $B_E$  is the base of  $E$  and  $B_F$  the base of  $F$ ,  $f(B_E) \subseteq B_F$ .*

*Proof.* Suppose  $x \in B_E$ , which is to say that  $x \in E_+$  and  $\tau(x) = 1$ . Since  $f$  is positive,  $f(x) \in F_+$ . By preservation of the trace, we have  $\sigma(f(x)) = \tau(x) = 1$ . Therefore  $f(x) \in B_F$ .  $\square$

The *sub-base* of a pre-base-norm space  $(E, E_+, \tau)$  is the set

$$B^{\leq 1} = E_+ \cap \tau^{-1}((-\infty, 1]).$$

**Lemma 2.2.8.** *We have three equivalent ways to express the sub-base:*

$$E_+ \cap \tau^{-1}((-\infty, 1]) = E_+ \cap \tau^{-1}([0, 1]) = \text{co}(\{0\} \cup B)$$

*Proof.*

- $E_+ \cap \tau^{-1}((-\infty, 1]) = E_+ \cap \tau^{-1}([0, 1])$ :

We have  $E_+ \cap \tau^{-1}([0, 1]) \subseteq E_+ \cap \tau^{-1}((-\infty, 1])$  immediately. By the positivity of  $\tau$ , if  $x \in E_+$ ,  $\tau(x) \geq 0$ , so the opposite inclusion also holds.

- $E_+ \cap \tau^{-1}([0, 1]) \subseteq \text{co}(\{0\} \cup B)$ :

Let  $x \in E_+ \cap \tau^{-1}([0, 1])$ . If  $x = 0$ ,  $x \in \text{co}(\{0\} \cup B)$ . Contrariwise, if  $x \neq 0$ , the strict positivity of  $\tau$  (Lemma 2.2.2) implies  $\tau(x) \neq 0$ , so we can take  $\frac{x}{\tau(x)} \in B$ . We may then express  $x$  in a manner clearly showing it is a convex combination in  $\text{co}(\{0\} \cup B)$ :

$$x = \tau(x) \frac{x}{\tau(x)} + (1 - \tau(x))0.$$

- $\text{co}(\{0\} \cup B) \subseteq E_+ \cap \tau^{-1}([0, 1])$ :

Since  $x \in \text{co}(\{0\} \cup B)$ , and  $\{0\}$  and  $B$  are both convex,  $x$  can be expressed as a convex combination

$$x = \alpha x' + (1 - \alpha)0,$$

and therefore that  $x = \alpha x'$  for  $x' \in B$ . This implies that  $x \in E_+$ . Now

$$\tau(x) = \tau(\alpha x') = \alpha \tau(x') = \alpha.$$

Since  $\alpha \in [0, 1]$ , this finishes the proof.  $\square$

**Corollary 2.2.9.** *For any pre-base-norm space  $(E, E_+, \tau)$  and  $\alpha \in \mathbb{R}_{>0}$ , we have  $E_+ \cap \alpha U = \alpha \text{co}(\{0\} \cup B)$ .*

*Proof.* We have

$$\begin{aligned} E_+ \cap \text{Ball}(\|\cdot\|_U) &= E_+ \cap \tau^{-1}([0, 1]) && \text{Corollary 2.2.5} \\ &= \text{co}(\{0\} \cup B) && \text{Lemma 2.2.8.} \end{aligned}$$

But this is not quite what we need. It is enough to show that

$$E_+ \cap U \subseteq E_+ \cap \text{Ball}(\|\cdot\|_U) = \text{co}(\{0\} \cup B),$$

using Lemma 0.1.6 for the first inclusion. Then, since  $E_+ \cap U$  is a convex set containing  $B$  and  $0$ , we have the other inclusion and so  $E_+ \cap U = \text{co}(\{0\} \cup B)$ .

If  $\alpha = 0$ , we have  $E_+ \cap \alpha U = \{0\} = \alpha \text{co}(\{0\} \cup B)$ . If, on the other hand,  $\alpha \neq 0$ , multiplying by  $\alpha$  is a bijection, so

$$\alpha \text{co}(\{0\} \cup B) = \alpha(E_+ \cap U) = \alpha E_+ \cap \alpha U = E_+ \cap \alpha U.$$

□

**Lemma 2.2.10.** *A trace-reducing map (and hence also a trace-preserving map) preserves the sub-base, i.e.  $f(B_E^{\leq 1}) \subseteq B_F^{\leq 1}$ .*

*Proof.* Let  $f : (X, X_+, \sigma) \rightarrow (Y, Y_+, \tau)$  be a trace-reducing map. Let  $x \in B_X^{\leq 1}$ , i.e.  $x \in X_+ \cap \tau^{-1}((-\infty, 1])$ . Since  $f$  is positive,  $f(x) \in Y_+$ . Since  $f$  is trace-reducing,  $\sigma(f(x)) \leq \tau(x) = 1$ , so  $f(x) \in \sigma^{-1}((-\infty, 1])$  as well. □

**Lemma 2.2.11.**  $\text{absco}(B) = \text{co}(-B^{\leq 1} \cup B^{\leq 1})$

*Proof.*

- $\text{absco}(B) \subseteq \text{co}(-B^{\leq 1} \cup B^{\leq 1})$ :

Consider  $x \in \text{absco}(B)$ , expressed as an absolutely convex combination:

$$x = \alpha_1 x_1 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n$$

with  $x_i \in B$ , and the indexing chosen so that  $\alpha_1, \dots, \alpha_k \geq 0$  and  $\alpha_{k+1}, \dots, \alpha_n \leq 0$ , with both sets of coefficients possibly empty (indicating the empty absolutely convex combination). Define

$$\beta_+ = \sum_{i=1}^k \alpha_i \quad \beta_- = \sum_{i=k+1}^n -\alpha_i.$$

These numbers are non-negative and  $\beta_+ + \beta_- \leq 1$ .

There are four possible cases, as each  $\beta$  can either be zero or nonzero. If  $\beta_+ = \beta_- = 0$ , then  $x = 0$ , so  $x \in B^{\leq 1} \subseteq \text{co}(-B^{\leq 1} \cup B^{\leq 1})$ . If one of them is nonzero, let  $s \in \{+, -\}$  be its sign. We have that

$$x = (1 - \beta_s)0 + \sum_i s\alpha_i x_i$$

is a convex combination, and so shows that  $x \in sB^{\leq 1} \subseteq \text{co}(-B^{\leq 1} \cup B^{\leq 1})$ .

Now suppose that  $\beta_+, \beta_- \neq 0$ . Define

$$x_+ = \sum_{i=1}^k \frac{\alpha_i}{\beta_+} x_i \quad x_- = \sum_{i=k+1}^n \frac{-\alpha_i}{\beta_-} x_i.$$

These are convex combinations, so  $x_+ \in B \subseteq B^{\leq 1}$  and  $x_- \in -B \subseteq -B^{\leq 1}$ . Let  $\beta_0 = 1 - \beta_+ - \beta_-$ . Define

$$x'_+ = \frac{\beta_0}{\beta_0 + \beta_+} 0 + \frac{\beta_+}{\beta_0 + \beta_+} x_+ \quad \beta'_+ = \beta_0 + \beta_+.$$

Now, by Lemma 2.2.8  $x'_+ \in B^{\leq 1}$ , and we have arranged it so that  $\beta'_+ + \beta_- = 1$  and they are both positive. Therefore  $\beta'_+ x'_+ + \beta_- x_- \in \text{co}(-B^{\leq 1} \cup B^{\leq 1})$  by definition. We have arranged it so that

$$\begin{aligned} \beta_+ x_+ + \beta_- x_- &= (\beta_0 + \beta_+) \left( \frac{\beta_0}{\beta_0 + \beta_+} 0 + \frac{\beta_+}{\beta_0 + \beta_+} x_+ \right) + \beta_- x_- \\ &= \beta_+ x_+ + \beta_- x_- \\ &= \beta_+ \left( \sum_{i=1}^k \frac{\alpha_i}{\beta_+} x_i \right) + \beta_- \left( \sum_{i=k+1}^n \frac{-\alpha_i}{\beta_-} x_i \right) \\ &= \sum_{i=1}^k \alpha_i x_i + \sum_{i=k+1}^n \alpha_i x_i = x. \end{aligned}$$

- $\text{co}(-B^{\leq 1} \cup B^{\leq 1}) \subseteq \text{absco}(B)$ :

We have that  $B \subseteq \text{absco}(B)$  and  $0 \subseteq \text{absco}(B)$ . Since convex combinations are a special case of absolutely convex combinations, Lemma 2.2.8 implies  $B^{\leq 1} \subseteq \text{absco}(B)$ . Since  $-1 \cdot x$  is an absolutely convex combination of  $x$ , we have that  $-B^{\leq 1} \subseteq \text{absco}(B)$  too. Reapplying the fact that convex combinations are a special case of absolutely convex combinations, we have that  $\text{co}(-B^{\leq 1} \cup B^{\leq 1}) \subseteq \text{absco}(B)$ .  $\square$

Note that the above identity holds even in the case that  $E = 0$ .

**Proposition 2.2.12.** *A trace-reducing morphism  $f : (E, E_+, \tau) \rightarrow (F, F_+, \sigma)$  of pre-base-norm spaces is bounded with operator norm  $\|f\| \leq 1$ . If  $f$  is trace-preserving and  $E \neq 0$ ,  $\|f\| = 1$ .*

*Proof.* Let  $f$  be trace-reducing. By applying Lemma 2.2.10, we have that  $f(B_E^{\leq 1}) \subseteq B_F^{\leq 1}$ . By Lemma 2.2.11, we have that  $\text{absco}(B_E) = \text{co}(-B_E^{\leq 1} \cup B_E^{\leq 1})$

and likewise for  $F$ . By linearity of  $f$ , we have that  $f(\text{co}(-B_E^{\leq 1} \cup B_E^{\leq 1})) \subseteq \text{co}(-B_F^{\leq 1} \cup B_F^{\leq 1})$ . The hypotheses of Lemma 0.1.8 are then satisfied, so we can conclude  $\|f\| \leq 1$ .

Now, if  $f$  is trace-preserving and  $E \neq 0$ , then by Lemma 2.2.1,  $B_E \neq \emptyset$ . So if  $x \in B_E$ , we have that  $x$  is positive and of trace 1, so by Corollary 2.2.5 we have  $\|x\| = 1$ . If  $x \in B_E$ , by Lemma 2.2.7  $f(x) \in B_F$ , and so  $\|f(x)\| = 1$  too. Therefore  $\|f\|$ , since it is an upper bound for  $\|f(x)\|$  as  $x$  varies over the closed unit ball of  $E$ , is greater than or equal to 1. Since  $\|f\| \leq 1$  in general, this shows that  $\|f\| = 1$ .  $\square$

We therefore have that  $f$  is continuous, and that there exist forgetful functors  $U_1 : \mathbf{PreBNS} \rightarrow \mathbf{Normed}_1$  and  $U_\infty : \mathbf{PreBNS} \rightarrow \mathbf{Normed}$ , where  $\mathbf{Normed}_1$  is the *metric* category of normed spaces, having maps of operator norm  $\leq 1$  (called *contractions*) as maps, and  $\mathbf{Normed}$  is the topological category of normed spaces, with bounded maps. These functors restrict to functors  $U_1 : \mathbf{BBNS} \rightarrow \mathbf{Ban}_1$  and  $U_\infty : \mathbf{BBNS} \rightarrow \mathbf{Ban}$ , where the  $\mathbf{Ban}_1$  and  $\mathbf{Ban}$  are full subcategories on Banach spaces.

## 2.2.2 Bounded Convex Sets

We define a category  $\mathbf{BConv}$  as follows. Its objects are pairs  $(E, X)$ , where  $E$  is a locally convex space and  $X \subseteq E$  is a subset that is bounded but also convex. The hom set is defined as

$$\mathbf{BConv}((E, X), (F, Y)) = \{f : X \rightarrow Y \mid f \text{ is affine}\}.$$

Note that we do not require the morphism to do anything with the ambient vector spaces, and we do not require any continuity for maps, the topology serves only to define boundedness. The purpose of this category is to package up a standard construction of a pre-base-norm space and morphisms between pre-base-norm spaces constructed in this manner. The following proposition is a version of [47, Theorem 2.2], although it is proved in a different way.

**Proposition 2.2.13.** *Let  $(E, X)$  be an object of  $\mathbf{BConv}$ . There exists a pre-base-norm space  $(F, F_+, \tau)$ , with a locally convex topology  $\mathcal{S}$  in which  $\tau$  is continuous, such that the topology and uniformity defined by the norm are finer than  $\mathcal{S}$  and its uniformity, and an isomorphism  $i : (E, X) \rightarrow (F, B_F)$  in  $\mathbf{BConv}$  that is a homeomorphism for the subspace topologies and a uniform isomorphism for the subspace uniformities on  $X$  and  $B_F$ .*

*Proof.* If  $X = \emptyset$ , let  $F$  be the zero base-norm space and take  $i = \text{id}_\emptyset$ . As there is only one topology and uniformity on the empty set, this is a uniform homeomorphism.

We now reduce to the case that  $X \neq \emptyset$ . Pick a point  $b \in X$ . We have that  $0 \in X - b$ , and we take  $E' = \text{span}(X - b)$ , giving it the subspace topology from  $E$ , which is locally convex. We then take  $F = \mathbb{R} \times E'$ , defining  $\mathcal{S}$  to be the locally convex product topology. We define

$$F_+ = \{\alpha(1, x) \mid \alpha \in \mathbb{R}_{\geq 0} \text{ and } x \in X - b\}$$

and  $\tau = \pi_1$ , which is therefore continuous. We must show that  $(F, F_+, \tau)$  is a pre-base-norm space. We first show that  $F_+$  is a cone generating  $F$ .

- $F_+$  closed under multiplication by  $\alpha \in \mathbb{R}_{\geq 0}$ :

This is immediately apparent from the definition above.

- $F_+$  is closed under addition:

Let  $\alpha(1, x)$  and  $\beta(1, y)$  be elements of  $F_+$ . Then

$$\begin{aligned} \alpha(1, x) + \beta(1, y) &= (\alpha + \beta, \alpha x + \beta y) \\ &= (\alpha + \beta) \left( 1, \frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \right), \end{aligned}$$

and we see that  $\alpha + \beta \in \mathbb{R}_{\geq 0}$ , and  $\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \in X - b$  because  $X - b$  is a convex subset of  $E'$ .

- $F_+ \cap -F_+ = \{0\}$ :

Let  $\alpha(1, x) = -\alpha(1, x)$ . Then in particular,  $\alpha = -\alpha$  so  $\alpha = 0$  and  $\alpha(1, x) = (0, 0)$ .

- $\text{span}(F_+) = F$ :

Let  $(\alpha, x) \in F$ . Since  $E' = \text{span}(X - x)$ , we have that  $x = \sum_{i=1}^n \alpha_i x_i$  for  $\alpha_i \in \mathbb{R}$  and  $x_i \in X - x$ . We define  $x_{n+1} = 0$ , as  $0 \in X - x$ , and  $\alpha_{n+1} = \alpha - \sum_{i=1}^n \alpha_i$ . Then

$$\sum_{i=1}^{n+1} \alpha_i = \alpha, \text{ and } \sum_{i=1}^{n+1} \alpha_i x_i = x + 0 = x,$$

so

$$\sum_{i=1}^{n+1} \alpha_i (1, x_i) = (\alpha, x)$$

and we have expressed it as a linear combination of elements of  $F_+$ .

We then need to show that  $\tau$  is nonzero and that  $\text{absco}(B_F)$  is radially bounded. Since  $0 \in X - b$ , we have  $(1, 0) \in F_+$ , and  $\tau(1, 0) = 1$ , so  $\tau \neq 0$ .

Now the base is

$$B_F = \{\alpha(1, x) | x \in X - b\} \cap \tau^{-1}(1),$$

and  $\tau(\alpha(1, x)) = 1$  implies that  $\alpha = 1$ , so

$$B_F = \{(1, x) | x \in X - b\}.$$

*i.e.*  $B_F = \{1\} \times (X - b)$ . By Lemma 0.1.13  $X - b$  is a bounded subset of  $E'$  and  $B_F$  is therefore a bounded subset of  $F$ . By Lemma 0.1.15  $\text{absco}(B_F)$  is bounded, and therefore radially bounded (Lemma 0.1.16). This shows that  $(F, F_+, \tau)$  is a pre-base-norm space.

To show that the pre-base-norm topology is finer than  $\mathcal{S}$ , let  $U$  be a 0-neighbourhood in  $\mathcal{S}$ . As  $\text{absco}(B_F)$  is bounded, there is an  $\alpha > 0$  such that  $\text{absco}(B_F) \subseteq \alpha U$ . By Lemma 0.1.6, the unit ball of  $F$  in its pre-base-norm topology is a subset of  $2\text{absco}(B_F)$ , so  $2\text{absco}(B_F)$  is a neighbourhood of zero. Since multiplication by a scalar is a homeomorphism, we have that  $2\alpha^{-1}\text{absco}(B_F)$  is a neighbourhood of zero, and therefore  $U$  is a neighbourhood of zero. Therefore every  $\mathcal{S}$ -open set is open in the pre-base-norm topology. Since the basic entourages for the uniformity are defined by  $\{(x, y) | x - y \in U\}$  for  $U$  a neighbourhood of 0, we have that the uniformity defined by the pre-base-norm is finer than the  $\mathcal{S}$ -uniformity.

We define  $i : X \rightarrow B_F$  as  $i(x) = (1, x - b)$ . We see that if  $x \in X$ , then the pair  $(1, x - b) \in \{1\} \times (X - b) = B_F$ , so  $i$  has the right type. We can decompose  $i$  as  $(- + (1, 0)) \circ \kappa_2 \circ (- + (-x))$ . The first and last part are affine uniform isomorphisms by Lemma 0.1.18, and the middle part is a linear homeomorphism, hence a uniform isomorphism, when restricted to  $E' \rightarrow \{0\} \times E'$  (Lemma 0.1.17). Therefore, when restricted to  $X \rightarrow B_F$ , it is an affine uniform isomorphism (and therefore a homeomorphism as well). It is also an isomorphism  $(E, X) \rightarrow (F, B_F)$  in **BConv**.  $\square$

Since the pre-base-norm topology is always finer than the original topology of a bounded convex set, it is the analogous notion for convex sets of the discrete topology on sets. The metric induced by the norm on  $B_E$  can be given an intrinsic definition, called  $\rho$  by [47, Theorem 3.2], but we do not require this here.

Given a pre-base-norm space  $(E, E_+, \tau)$ , we have seen that we can define an element of **BConv** as  $(E, B_E)$ , taking the locally convex topology to be that defined by the norm. Lemma 2.2.7 implies that if we have a trace-preserving



morphism  $f : (E, E_+, \tau) \rightarrow (F, F_+, \sigma)$  then  $f|_{B_E}$  restricts to have codomain  $B_F$ , and so  $f|_{B_E}$  is therefore a map  $(E, B_E) \rightarrow (F, B_F)$  in **BConv**. This defines a functor  $B : \mathbf{PreBNS} \rightarrow \mathbf{BConv}$ , which is essentially surjective by Proposition 2.2.13, and is faithful by definition.

**Lemma 2.2.14.** *Let  $(E, E_+, \tau)$  be a pre-base-norm space,  $\mathcal{T}$  a locally convex topology on  $E$  such that  $\tau$  is sequentially continuous. Then  $B_E$  is sequentially closed iff  $E_+$  is sequentially closed.*

*Proof.* If  $E_+$  is sequentially closed, then as  $\tau^{-1}(1)$  is sequentially closed, we have  $B_E = E_+ \cap \tau^{-1}(1)$  is sequentially closed.

For the other direction, suppose that  $B_E$  is sequentially closed, and let  $(x_i)$  be a sequence in  $E_+$  converging in  $\mathcal{T}$  to  $x \in E$ . To show  $E_+$  is sequentially closed we must show that  $x$  is in  $E_+$ . As  $\tau$  is sequentially continuous,  $\tau(x_i)$  converges to  $\tau(x)$ . If  $\tau(x) = 0$ , then  $x = 0$  by Lemma 2.2.2, so  $x_i \rightarrow 0$ , which is an element of  $E_+$ . We can therefore reduce to the case that  $\tau(x) > 0$ . Define  $(y_i)$  to be the subsequence of  $(x_i)$  starting at the  $n$  such that for all  $i \geq n$   $|\tau(x_i) - \tau(x)| < \frac{\tau(x)}{2}$ , a value that must exist by the convergence of  $(\tau(x_i))_i$ . We therefore have  $y_i \rightarrow x$  and  $\tau(y_i) > 0$  for all  $i \in \mathbb{N}$ . Define  $z_i = \frac{y_i}{\tau(y_i)}$  and  $z = \frac{x}{\tau(x)}$ . Since  $^{-1} : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}$  is continuous, we have  $\frac{1}{\tau(y_i)} \rightarrow \frac{1}{\tau(x)}$ . By joint continuity of scalar multiplication, we have  $\frac{y_i}{\tau(y_i)} \rightarrow \frac{x}{\tau(x)}$ , i.e.  $z_i \rightarrow z$ . Because  $z_i$  is a sequence in  $B_E$ , we have  $z \in B_E$ , and therefore  $x = \tau(x)z \in E_+$ .  $\square$

The following lemma is based on [110, V.3.4 Lemma 2] and is stated in this way because it will be used later in two different proofs.

**Lemma 2.2.15.** *Let  $(E, \|\cdot\|)$  be a normed space,  $U = \text{Ball}(\|\cdot\|)$ , and  $E_+ \subseteq E$  be a cone such that  $E_+ \cap U$  is  $\sigma$ -convex. Define  $F = E_+ - E_+$  and take  $V_1 = \text{co}(E_+ \cap U \cup -E_+ \cap U)$ ,  $V_2 = E_+ \cap U - E_+ \cap U$ . Then  $V_1$  and  $V_2$  define equivalent norms  $\|\cdot\|_{V_1}, \|\cdot\|_{V_2}$  on  $F$  in which it is complete.*

*Proof.* We first show that  $V_1$  and  $V_2$  define equivalent norms. We can see that  $V_1 \subseteq V_2$  as follows. If  $\alpha x_+ - (1 - \alpha)x_- \in V_1$ , i.e.  $x_+, x_- \in U \cap E_+$  and  $\alpha \in [0, 1]$ , then  $\alpha x_+$  and  $(1 - \alpha)x_-$  are elements of  $U \cap E_+$  by absolute convexity of  $U$  and  $E_+$  being a cone. Therefore  $\alpha x_+ - (1 - \alpha)x_- \in E_+ \cap U - E_+ \cap U = V_2$ .

We then show that  $V_2 \subseteq 2V_1$  as follows. If  $x_+ - x_- \in V_2$ , which is to say,  $x_+, x_- \in E_+ \cap U$ , then  $x_+ - x_- = 2(\frac{1}{2}x_+ - \frac{1}{2}x_-) \in 2\text{co}(E_+ \cap U \cup -E_+ \cap U)$ .

Both  $V_1$  and  $V_2$  are clearly balanced as their definitions are equivalent when negated. Then  $V_1$  is convex by its definition as a convex hull, while for  $V_2$ , if

we have  $x_+ - x_-, y_+ - y_- \in V_2$ , and  $\alpha \in [0, 1]$ , then

$$\alpha(x_+ - x_-) + (1 - \alpha)(y_+ - y_-) = (\alpha x_+ + (1 - \alpha)y_+) - (\alpha x_- + (1 - \alpha)y_-) \in V_2$$

by the convexity of  $E_+ \cap U$ . We also have that  $0 \in V_1$  and  $0 \in V_2$ , so neither of the sets is empty, so  $V_1$  and  $V_2$  are absolutely convex by Lemma A.3.1.

The containment results between  $V_1 \subseteq V_2$  and  $V_2 \subseteq 2V_1$  show that each is absorbent iff the other is, so we show that  $V_2$  is absorbent (in  $F$ ). Let  $x_+ - x_- \in E_+ - E_+ = F$ . As  $U$  is absorbent, being the unit ball of a norm, there exist  $\alpha, \beta \in \mathbb{R}_{>0}$  such that  $x_+ \in \alpha U$  and  $x_- \in \beta U$ , and these also hold for any greater real number in either case. Therefore, if we take  $\gamma = \max\{\alpha, \beta\}$ , we conclude that  $x_+, x_- \in \gamma U$ , so  $x_+ - x_- \in E_+ \cap \gamma U - E_+ \cap \gamma U = \gamma V_2$ .

This proves that  $\|\cdot\|_{V_1}$  and  $\|\cdot\|_{V_2}$  are norms, and are equivalent. Therefore  $F$  is complete in one iff it is complete in the other, so we show that  $F$  is complete in  $\|\cdot\|_{V_2}$ . Let  $(a_i)_{i \in \mathbb{N}}$  be a  $\|\cdot\|_{V_2}$ -Cauchy sequence in  $F$ . We can select a subsequence<sup>2</sup>  $(b_i)$  such that  $b_{i+1} - b_i \in 2^{-i}V_2$  for all  $i \in \mathbb{N}$ . Therefore there exist  $x_i, y_i \in E_+ \cap 2^{-i}U$  such that  $b_{i+1} - b_i = x_i - y_i$ .

We can define  $x'_i = 2^i x_i$  and  $y'_i = 2^i y_i$ , and these are sequences in  $E_+ \cap U$ . Then  $\sum_{i=1}^n x_i = \sum_{i=1}^n 2^{-i} x'_i$ , so by the  $\sigma$ -convexity of  $E_+ \cap U$ ,  $\sum_{i=1}^\infty x_i$  converges, as does  $\sum_{i=1}^\infty y_i$ . We call these sums  $x$  and  $y$  respectively.

Then we can define  $b = x - y + b_1 \in F$ . To finish the proof that  $F$  is complete, we will show that  $(b_i)$ , and therefore  $(a_i)$ , converges to  $b$ . For any given  $\epsilon > 0$ , there exist  $j$  and  $k$  such that  $\left\|x - \sum_{i=1}^j x_i\right\|_{V_2} < \frac{\epsilon}{2}$  and  $\left\|y - \sum_{i=1}^k y_i\right\|_{V_2} < \frac{\epsilon}{2}$ . If we take  $m = \max\{j, k\}$ , then for all  $n \geq m$  we have that

$$\left\|\left(x - \sum_{i=1}^n x_i\right) + \left(y - \sum_{i=1}^n y_i\right)\right\|_{V_2} < \epsilon, \text{ so } \left\|(x - y) - \sum_{i=1}^n (x_i - y_i)\right\|_{V_2} < \epsilon.$$

Then

$$\sum_{i=1}^n (x_i - y_i) = \sum_{i=1}^n (b_{i+1} - b_i) = b_{n+1} - b_1,$$

as it is a telescoping sum. We can therefore conclude that

$$\|(x - y + b_1) - (b_{n+1} - b_1 + b_1)\|_{V_2} < \epsilon,$$

so  $\|b - b_{n+1}\|_{V_2} < \epsilon$ . Therefore for all  $n \geq m + 1$  we have  $\|b - b_n\|_{V_2} < \epsilon$ , and so  $(b_i)$  converges to  $b$  in  $\|\cdot\|_{V_2}$ .  $\square$

<sup>2</sup>By taking  $b_i$  to be the  $a_N$  where  $N$  is the smallest number such that for all  $j, k \geq N$   $\|a_j - a_k\|_{V_2} < 2^{-i}$ , which necessarily exists for a Cauchy sequence.

### 2.2.3 Comparison of Definitions

There are other definitions of base-norm space available in the literature. Some of the differences are superficial, such as using a hyperplane to define the base of the cone instead of the linear functional  $\tau$ , as in [4, §II.1 p. 77] [6, p. 9]. There is also the definition used by Nagel [88, §2], which is the same as ours except requiring that  $\tau$  be *strictly* positive and relaxing radially compact to radially bounded. In Asimow and Ellis's definition [8, Definition, p.36] a base-norm space is defined to be a particular kind of normed ordered vector space (with closed positive cone).

In the following, we will give a proof that the Alfsen-Shultz definition coincides with radially compact, non-zero base-norm spaces, a proof that Nagel's definition coincides with pre-base-norm spaces, and two counterexamples – a pre-base-norm space that is not a base-norm space, and a base-norm space that is not radially compact. Asimow and Ellis's definition agrees with ours, except in the case of the zero base-norm space, but we leave the proof of this as an exercise to the reader.

#### Nagel's Definition

Nagel's definition [88, §2] is that a base-norm space is a triple  $(E, E_+, \tau)$ , where  $(E, E_+)$  is a directed ordered vector space,  $\tau$  is a strictly positive linear functional  $\tau : E \rightarrow \mathbb{R}$ , and with  $B$  having its usual definition,  $U = \text{co}(-B \cup B)$  is radially bounded. Because of the use of  $U = \text{co}(-B \cup B)$  rather than  $U = \text{absc}(B)$ , the zero base-norm space does not satisfy Nagel's definition. However, by Lemma 2.2.2 every non-zero pre-base-norm space is a base-norm space in Nagel's sense, and every base-norm space in Nagel's sense is a pre-base-norm space.

However, it is not the case that every pre-base-norm space is a base-norm space. We construct a counter-example as follows, which we call the *strict plane*. Take the underlying vector space to be  $E = \mathbb{R}^2$ . We take the positive cone  $E_+$  to be

$$E_+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \cup \{(0, 0)\}.$$

This is a cone, as it can be seen to satisfy the axioms by elementary manipulation of inequalities. We show that  $E_+$  is generating as follows. Let  $(x, y) \in \mathbb{R}^2$ . Each real number  $z$  can be expressed as the difference of two strictly positive numbers as follows. Pick some  $\epsilon > 0$ . If  $z > 0$ ,  $z_+ = z + \epsilon$  and  $z_- = \epsilon$ . If  $z = 0$ , take  $z_+ = \epsilon$  and  $z_- = \epsilon$ . If  $z < 0$ , take  $z_+ = \epsilon$  and  $z_- = -z + \epsilon$ . Apply this decomposition independently to  $x$  and  $y$ , and we have

$(x, y) = (x_+, y_+) - (x_-, y_-)$ . Since their components are strictly positive, we have that  $(x_+, y_+)$  and  $(x_-, y_-) \in E_+$ , as required. So  $(E, E_+)$  is a directed partially ordered vector space.

We define the trace  $\tau : E \rightarrow \mathbb{R}$  as  $\tau(x, y) = x + y$ . We can see this is positive and not zero.

We therefore only need to show that  $U = \text{co}(B \cap -B)$  is radially bounded.

**Lemma 2.2.16.**  *$U$  is radially bounded.*

*Proof.* We show this by showing that  $B$  is contained in the closed unit ball  $D$  for the Hilbert space norm on  $\mathbb{R}^2$ , which is the unit sphere. Since  $U = \text{absco}(B)$  and  $D$  is absolutely convex, we can conclude  $U \subseteq D$ . Then radial boundedness follows from the radial boundedness of  $D$ , which comes from the fact that  $D$  contains no line through the origin (using Lemma 0.1.2).

We have that

$$\begin{aligned} B &= E_+ \cap \tau^{-1}(1) \\ &= (\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \cup \{(0, 0)\}) \cap \{(x, y) \in \mathbb{R}^2 \mid x + y = 1\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0, x + y = 1\}. \end{aligned}$$

We need to show that  $x > 0$ ,  $y > 0$ , and  $x + y = 1$  implies  $x^2 + y^2 \leq 1$ . We have that  $1 = 1^2 = (x + y)^2 = x^2 + 2xy + y^2$ . We also have that  $2xy > 0$ , using the two inequalities. Therefore

$$x^2 + y^2 = 1 - 2xy < 1$$

as required. By the previous paragraph, this is enough to prove the lemma.  $\square$

We have now proved that  $(E, E_+, \tau)$  is a pre-base-norm space, or equivalently a base-norm space in Nagel's sense.

**Counterexample 2.2.17.**  *$(E, E_+, \tau)$  is not a base-norm space.*

*Proof.* By [110, Theorem I.3.2] the norm topology on  $E$  agrees with the usual topology on  $\mathbb{R}^2$  as it is Hausdorff and finite-dimensional. Therefore we can see that  $E_+$  is not closed in  $E$ , so  $(E, E_+, \tau)$  is not a base-norm space.  $\square$

At first, this might seem like a deficiency of the definition we use. But a space with an unclosed positive cone can never occur as a dual cone because dual cones are always closed (Lemma 0.3.7 and Theorem 0.3.9). This makes it reasonable to restrict to those spaces with closed positive cones. Pre-base-norm spaces still have their advantages in certain situations, however, such as when one wants to find a space having a given convex set as its base, as we saw in Subsection 2.2.2.

### The Alfsen-Shultz Definition

This definition can be found in [4, §II.1 p. 77] [6, p. 9]. We repeat it here. It depends on the definition of the base of a cone, which is found in [6, p. 3][4, p. 76], which we define first. Given a cone  $E_+ \subseteq E$ , a base  $B \subseteq E_+$  is the convex set given by the intersection of a hyperplane  $H$  ( $0 \notin H$ ) with  $E_+$ , *i.e.*  $B = E_+ \cap H$ , subject to the additional requirement that  $E_+ = \bigcup_{\alpha \geq 0} \alpha B$ . A base-norm space is then directed ordered vector space  $(E, E_+)$  and a choice of base  $B$  for  $E_+$ , such that

$$U = \text{co}(B \cup -B)$$

is radially compact.

#### Proposition 2.2.18.

- (i) *The base of a base-norm space in the Alfsen-Shultz sense cannot be empty.*
- (ii) *The base-norm space (in our sense)  $(\{0\}, \{0\}, 0)$ , is not a base-norm space in the Alfsen-Shultz sense.*

*Proof.*

- (i) Suppose  $B = \emptyset$ . Then  $E_+ = \bigcup_{\alpha \geq 0} \alpha B = \emptyset$ , and so  $E = E_+ - E_+ = \emptyset$ , which contradicts  $E$  being a vector space, as it must contain 0.
- (ii) It is clear that since the base of  $\{0\}$  is empty it cannot be an Alfsen-Shultz base-norm space with the same base. In fact, it cannot be one at all, because it contains no hyperplanes.  $\square$

**Proposition 2.2.19.** *Every non-zero radially compact pre-base-norm space is an Alfsen-Shultz base-norm space, with the same base and unit ball.*

*Proof.* By definition,  $\tau^{-1}(1)$  is a hyperplane, and  $B = E_+ \cap \tau^{-1}(1)$ . Since  $\tau(0) = 0$ , we have  $0 \notin \tau^{-1}(1)$ . Since  $\text{co}(B \cup -B)$  is radially compact, we only need to show that  $B$  is actually a base for  $E_+$ , *i.e.* that  $E_+ = \bigcup_{\alpha \geq 0} \alpha B$ .

Let  $x \in E_+$ . We start with the case that  $x = 0$ . Since  $E \neq 0$ , we have that there is  $x' \in B$  (Lemma 2.2.1), and therefore  $0 \cdot x' = x \in 0 \cdot B$ . If  $x \neq 0$ , using strict positivity (Lemma 2.2.2) we have  $\tau(x) > 0$ . Therefore we can define  $x' = \frac{x}{\tau(x)}$ , and  $\tau(x') = 1$ , so  $x' \in B$ , and therefore  $x \in \tau(x)B$ . We have proven that  $E_+ \subseteq \bigcup_{\alpha \geq 0} \alpha B$ . The opposite inclusion follows from the fact that  $E_+$  is a cone and  $B \subseteq E_+$ .  $\square$

**Proposition 2.2.20.** *Every base-norm space in the Alfsen-Shultz sense is a radially compact base-norm space, with the same base and unit ball.*

*Proof.* We must define  $\tau$ . We have, by Proposition 2.2.18 (i) that the base  $B$  is not empty, so there is  $x \in B$ . By taking the hyperplane  $H$ , and producing  $H - x$ , we have a hyperplane passing through 0, and therefore  $E/(H - x) \cong \mathbb{R}$ , so we can define a map  $\tau' : E \rightarrow \mathbb{R}$  by composing the surjection  $E \rightarrow E/(H - x)$  with the isomorphism with  $\mathbb{R}$ . If  $\tau'(x) = 0$ , then  $0 - x \in H - x$  and hence  $0 \in H$ , a contradiction, so  $\tau'(x) \neq 0$ . We can therefore take  $\tau = \frac{\tau'}{\tau'(x)}$ . We have shown that  $\tau$  is not zero.

We next show that  $B = E_+ \cap \tau^{-1}(1)$ , and prove that  $\tau$  is positive last. If  $y \in E_+$  and  $\tau(y) = 1$ , then  $\tau(y - x) = 0$ , so  $y - x \in H - x$ , therefore  $y \in H$ , and so  $y \in H \cap E_+ = B$ . This shows  $E_+ \cap \tau^{-1}(1) \subseteq B$ . For the opposite inclusion, if  $y \in B$ , then  $y \in H$  and so  $y - x \in H - x$ , meaning  $\tau(y - x) = 0$ . Therefore  $\tau(y) = \tau(x) = 1$ , and therefore  $y \in E_+ \cap \tau^{-1}(1)$ .

To show that  $\tau$  is positive, suppose  $y \in E_+$ . Because  $B$  is a base, there exists some  $\alpha \geq 0$  such that  $y \in \alpha B$ . This means that there is some  $y' \in B$  such that  $y = \alpha y'$ . We have

$$\tau(y) = \tau(\alpha y') = \alpha \tau(y') = \alpha \geq 0,$$

using the previous result that  $B = E_+ \cap \tau^{-1}(1)$  to make the penultimate step.

Since  $B$  is non-empty, the radial compactness of  $\text{co}(B \cup -B)$  implies that of  $\text{absco}(B)$ , as they are equal by Lemma 0.1.1. We have therefore shown that  $(E, E_+, \tau)$  is a radially compact pre-base-norm space. It is therefore a base-norm space by Proposition 2.2.6 (ii).  $\square$

All together, this shows that, except for the zero base-norm space, Alfsen and Shultz's definition of a base-norm space is at least as strict as ours, because it coincides with radially compact base-norm spaces. In the appendix we give a counterexample (Counterexample A.6.2) due to Asimow but published by Ellis [35] of a Banach base-norm space such that  $U = \text{absco}(B)$  is not radially compact. Therefore, for nonzero vector spaces, Alfsen and Shultz's definition is stricter.

## 2.3 Relationship to $C^*$ and $W^*$ -algebras

In Proposition 1.2.10 we saw that taking the self-adjoint part of a  $C^*$ -algebra yields a full and faithful functor to the category of Banach order-unit spaces. This, in fact, is one of the motivations for the definition of an order-unit space. In this section, we describe another full and faithful functor and the kind of space that motivated the definition of a base-norm space.

A  $W^*$ -algebra is a  $C^*$ -algebra  $A$  that is isometric<sup>3</sup> to the dual space of some Banach space  $A_*$  [109, Definition 1.1.2][118, Theorem 3.5]. Equivalently, it is a  $C^*$ -algebra  $A$  such that there exists a Banach space  $A_*$  and a duality  $\langle -, - \rangle : A \times A_* \rightarrow \mathbb{C}$  such that the map  $A \rightarrow (A_*)^*$  defined by the duality is an isometry.  $W^*$ -algebras were defined by Sakai to give a characterization of the  $C^*$ -algebras arising from von Neumann algebras up to isomorphism. The space  $A_*$  is called the *predual* and is unique up to isomorphism [109, Corollary 1.13.3][118, Corollary 3.9]. The urexample of a  $W^*$ -algebra and its predual are  $B(\mathcal{H})$ , the  $C^*$ -algebra of all bounded operators on a Hilbert space  $\mathcal{H}$ , and its predual  $\mathcal{TC}(\mathcal{H})$ , the space of trace-class operators, the pairing being

$$\langle a, \rho \rangle = \text{tr}(a\rho),$$

where  $a \in B(\mathcal{H})$  and  $\rho \in \mathcal{TC}(\mathcal{H})$ <sup>4</sup>. In this case, one can define self-adjoint and positive elements of  $\mathcal{TC}(\mathcal{H})$  in the usual way as  $\mathcal{TC}(\mathcal{H}) \subseteq B(\mathcal{H})$ , and the trace  $\tau(\rho)$  of a trace-class operator  $\rho$  can be defined as the sum of the diagonal entries of a matrix for  $\rho$ , expressed in some orthonormal basis<sup>5</sup>. The convex set

$$\mathcal{DM}(\mathcal{H}) = \{\rho \in \mathcal{TC}(\mathcal{H}) \mid \rho \text{ positive and } \tau(\rho) = 1\},$$

is known as the set of density matrices. So we have the ingredients for a base-norm space with base  $\mathcal{DM}(\mathcal{H})$ .

We first discuss (continuous) linear functionals on a  $C^*$ -algebra  $A$ , or elements of  $A^*$ . The involution  $-^* : A \rightarrow \bar{A}$  can be used to define an involution on  $A^*$ :

$$\phi^*(a) = \overline{\phi(a^*)},$$

where  $a \in A$  and  $\phi \in A^*$  [26, §1.1.10]. A functional  $\phi : A \rightarrow \mathbb{C}$  is therefore *self-adjoint* if  $\phi^* = \phi$ . By taking the complex conjugate, this is equivalent to  $\phi(a^*) = \overline{\phi(a)}$ , i.e.  $\phi$  maps the  $-^*$  operation to complex conjugation.

In the case of a general  $W^*$ -algebra  $A$ , we can embed the predual  $A_*$  isometrically into  $A^*$  by transposing the duality between  $A_*$  and  $A$ . We can use the freedom of choosing the predual up to isomorphism to redefine it to be this subset of  $A^*$ . It can equivalently be defined, by Proposition 0.3.2 to be the elements of  $A^*$  that are  $\sigma(A, A_*)$ -continuous. These are known as *normal linear functionals*, and the  $\sigma(A, A_*)$  topology is called the *ultraweak* or  *$\sigma$ -weak*

<sup>3</sup>Isomorphism, in the sense of a linear homeomorphism, is not sufficient [120].

<sup>4</sup>In the finite dimensional case, every operator is trace-class.

<sup>5</sup>Independence of  $\tau(\rho)$  from the orthonormal basis chosen is essentially what being of trace-class amounts to.

topology. One can therefore define  $\text{SA}(A_*)$  to be the  $\mathbb{R}$ -vector space of self-adjoint elements of  $A_*$ , and  $A_{*+}$  to be the set of positive elements of  $A_*$ . We have a linear map  $\tau : \text{SA}(A_*) \rightarrow \mathbb{R}$  defined as  $\tau(\phi) = \phi(1)$ .

**Theorem 2.3.1.** *If  $A_*$  is the predual of a  $W^*$ -algebra,  $(\text{SA}(A_*), A_{*+}, \tau)$  is a (radially compact) Banach base-norm space. If **Predual** is the category having preduals as objects and linear, positive, trace-preserving maps, restriction of morphisms defines a full and faithful functor  $\text{SA} : \mathbf{Predual} \rightarrow \mathbf{BBNS}$ .*

*Proof.* The fact that the self-adjoint part of the predual of a  $W^*$ -algebra is proven in [31, Proposition 5.1] and in [6, Corollary 2.96]. The definitions of  $W^*$ -algebra and base-norm space used in those references exclude the  $W^*$ -algebra in which  $0 = 1$ , but the self-adjoint part of the predual is the unique base-norm space with empty base in this case, so there is no problem. Note that this implies that the real span of  $A_{*+}$  is  $\text{SA}(A_*)$ , so if  $f : A_* \rightarrow B_*$  is a map of preduals, then if  $\phi \in \text{SA}(A_*)$ , we have  $f(\phi) \in \text{SA}(B_*)$ . Preservation of identity and composition for the functor  $\text{SA}$  is then trivial. Analogously to Lemma 1.2.2, elements of the predual have a decomposition into real and imaginary self-adjoint parts with

$$\phi_{\Re} = \frac{\phi + \phi^*}{2} \qquad \phi_{\Im} = \frac{\phi - \phi^*}{2i}$$

and the proof of fullness and faithfulness proceeds along the same lines as Proposition 1.2.10, so it is omitted.  $\square$

The base of  $\text{SA}(A_*)$  is the set of states that are  $\sigma(A, A_*)$  continuous as maps  $A \rightarrow \mathbb{C}$ , and is accordingly known as the set of *normal states*. In the special case that  $A = B(\mathcal{H})$ , this is  $\mathcal{DM}(\mathcal{H})$ , as would be expected.

In the next chapter, we will prove a statement implying that the “predual” of any order-unit space is a base-norm space, giving a proof that the self-adjoint part of the predual of a  $W^*$ -algebra is a base-norm space independently of the results cited above.

## 2.4 Relationship to Monads

The monads are  $\mathcal{D}, \mathcal{D}^{\leq 1}, \mathcal{D}_{\infty}$  and  $\mathcal{D}_{\infty}^{\leq 1}$ , all functors  $\mathbf{Set} \rightarrow \mathbf{Set}$ . The monad  $\mathcal{D}$  is the usual distribution monad,  $\mathcal{D}^{\leq 1}$  the subnormalized version,  $\mathcal{D}_{\infty}$  the infinite distribution monad, and  $\mathcal{D}_{\infty}^{\leq 1}$  its subnormalized version. We summarize the definitions here, but do not prove they are monads as that is adequately explained elsewhere. Apparently the idea of using infinite convex combinations



on state spaces is due to Michael A. Gerzon [31, p. 214], and later appeared under the name *superconvex sets* [104], see also [73] and [74].

On objects, the functors are defined:

$$\begin{aligned}\mathcal{D}(X) &= \left\{ \phi : X \rightarrow [0, 1] \left| \begin{array}{l} \text{supp}(\phi) \text{ finite and } \sum_{x \in X} \phi(x) = 1 \end{array} \right. \right\} \\ \mathcal{D}^{\leq 1}(X) &= \left\{ \phi : X \rightarrow [0, 1] \left| \begin{array}{l} \text{supp}(\phi) \text{ finite and } \sum_{x \in X} \phi(x) \leq 1 \end{array} \right. \right\} \\ \mathcal{D}_{\infty}(X) &= \left\{ \phi : X \rightarrow [0, 1] \left| \sum_{x \in X} \phi(x) = 1 \right. \right\} \\ \mathcal{D}_{\infty}^{\leq 1}(X) &= \left\{ \phi : X \rightarrow [0, 1] \left| \sum_{x \in X} \phi(x) \leq 1 \right. \right\}.\end{aligned}$$

On a map  $f : X \rightarrow Y$  in **Set**, we give the formula for  $\mathcal{D}$  only, as it is the same for the other three. Let  $\phi \in \mathcal{D}(X)$  and  $y \in Y$ :

$$\mathcal{D}(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x).$$

The unit and counit are defined the same for all four monads, so we give the definition only for  $\mathcal{D}$ :

$$\begin{aligned}\eta_X : X &\rightarrow \mathcal{D}(X) \\ \eta_X(x)(x') &= 1 \text{ if } x = x' \\ \eta_X(x)(x') &= 0 \text{ otherwise} \\ \mu_X : \mathcal{D}^2(X) &\rightarrow \mathcal{D}(X) \\ \mu_X(\Psi)(x) &= \sum_{\phi \in \mathcal{D}(X)} \Psi(\phi) \cdot \phi(x).\end{aligned}$$

There are monad morphisms  $\tau : \mathcal{D} \Rightarrow \mathcal{D}_{\infty}$  and  $\tau^{\leq 1} : \mathcal{D}^{\leq 1} \Rightarrow \mathcal{D}_{\infty}^{\leq 1}$ .

**Proposition 2.4.1.** *The family of maps  $\tau_X : \mathcal{D}(X) \rightarrow \mathcal{D}_{\infty}(X)$  taking the finite distributions into the infinite ones is natural and a monad morphism. The same is true for  $\tau_X^{\leq 1} : \mathcal{D}^{\leq 1}(X) \rightarrow \mathcal{D}_{\infty}^{\leq 1}(X)$ .*

*Proof.* The definition of  $\tau_X$  is

$$\tau_X(\phi) = \phi.$$

This is clearly natural. The definition of  $\tau_X^{\leq 1}$  is identical.

In the following we only give the proof for  $\tau_X$  as the proof for  $\tau_X^{\leq 1}$  is identical as the definitions of the maps involved coincide.

The triangle

$$\begin{array}{ccc} I & \xrightarrow{\eta^{\mathcal{D}}} & \mathcal{D} \\ & \searrow \eta_{\infty}^{\mathcal{D}} & \downarrow \tau \\ & & \mathcal{D}_{\infty} \end{array}$$

commutes as  $\eta_X^{\mathcal{D}_{\infty}}(x) \in \mathcal{D}(X)$  and  $\tau_X$  is just the inclusion morphism. The pentagon

$$\begin{array}{ccc} \mathcal{D}^2 & \xrightarrow{\mathcal{D}\tau} & \mathcal{D}\mathcal{D}_{\infty} \\ \mu^{\mathcal{D}} \downarrow & & \downarrow \tau \mathcal{D}_{\infty} \\ \mathcal{D} & & \mathcal{D}_{\infty}^2 \\ & \searrow \tau & \downarrow \mu_{\infty}^{\mathcal{D}} \\ & & \mathcal{D}_{\infty} \end{array}$$

can be proved to commute as follows. Let  $\Phi \in \mathcal{D}^2(X)$ , and  $x \in X$ . For the lower left path we have

$$\tau_X(\mu_X^{\mathcal{D}}(\Phi))(x) = \mu_X^{\mathcal{D}}(\Phi)(x) = \sum_{\psi \in \mathcal{D}(X)} \Phi(\psi) \cdot \psi(x)$$

and for the upper right path we have

$$\begin{aligned} \mu_X^{\mathcal{D}_{\infty}}(\tau_{\mathcal{D}_{\infty}(X)}(\mathcal{D}(\tau_X)(\Phi)))(x) &= \sum_{\psi \in \mathcal{D}_{\infty}(X)} \tau_{\mathcal{D}_{\infty}(X)}(\mathcal{D}(\tau_X)(\Phi))(\psi) \cdot \psi(x) \\ &= \sum_{\psi \in \mathcal{D}_{\infty}(X)} \mathcal{D}(\tau_X)(\Phi)(\psi) \cdot \psi(x) \\ &= \sum_{\psi \in \mathcal{D}_{\infty}(X)} \left( \sum_{\phi \in \tau_X^{-1}(\psi)} \Phi(\phi) \right) \cdot \psi(x) \\ &= \sum_{\psi \in \mathcal{D}_{\infty}(X)} \sum_{\phi \in \tau_X^{-1}(\psi)} \Phi(\phi) \cdot \psi(x) \end{aligned}$$

The inner sum consists of one term if  $\psi \in \mathcal{D}(X)$ , and is zero if  $\psi$  has infinite support, so effectively the inner sum restricts us to summing over  $\mathcal{D}(X)$  instead of  $\mathcal{D}_\infty(X)$ . Therefore we have

$$= \sum_{\psi \in \mathcal{D}(X)} \Phi(\psi) \cdot \psi(x)$$

also for the top right path and the diagram commutes.  $\square$

By Proposition 0.4.8, these monad morphisms imply the existence of forgetful functors  $\mathcal{EM}(\mathcal{D}_\infty) \rightarrow \mathcal{EM}(\mathcal{D})$  and  $\mathcal{EM}(\mathcal{D}^{\leq 1}) \rightarrow \mathcal{EM}(\mathcal{D}_\infty^{\leq 1})$ .

### 2.4.1 The Base and Subbase Functors and Their Left Adjoints

We saw earlier what the base  $B_E$  of a pre-base-norm space  $(E, E_+, \tau)$  is and the subbase  $B_E^{\leq 1}$ . We can define two functors

$$B_{\mathbf{Set}} : \mathbf{PreBNS} \rightarrow \mathbf{Set} \quad B_{\mathbf{Set}}^{\leq 1} : \mathbf{PreBNS}_{\leq 1} \rightarrow \mathbf{Set},$$

on objects being the base and the subbase. On maps, these are simply restriction, which is well defined by Lemmas 2.2.7 and 2.2.10 respectively. By restriction to the full subcategories  $\mathbf{BBNS}$  and  $\mathbf{BBNS}_{\leq 1}$  we also have functors from those categories to  $\mathbf{Set}$ . We now define their left adjoints  $\ell_c^1 : \mathbf{Set} \rightarrow \mathbf{PreBNS}$  and  $\ell^1 : \mathbf{Set} \rightarrow \mathbf{BBNS}$ . We define these on a set  $X$ , as

$$\begin{aligned} \ell_c^1(X) &= \{\phi : X \rightarrow \mathbb{R} \mid \text{supp}(\phi) \text{ is finite}\} \\ \ell^1(X) &= \left\{ \phi : X \rightarrow \mathbb{R} \mid \sum_{x \in X} |\phi(x)| < \infty \right\}. \end{aligned}$$

The vector space structure is defined pointwise, and it is clear that  $\ell_c^1(X)$  is a subspace of  $\ell^1(X)$ . We define the positive cone in each to be those  $\phi$  such that  $\phi(x) \geq 0$  for all  $x \in X$ , and we define the trace to be

$$\tau(\phi) = \sum_{x \in X} \phi(x),$$

which exists for all  $\phi \in \ell^1(X)$  by Lemma 0.1.11.

It will be useful later to observe that for each  $\phi \in \ell^1(X)$ , we can separate it into its positive and negative parts:

$$\phi_+(x) = \begin{cases} \phi(x) & \text{if } \phi(x) > 0 \\ 0 & \text{otherwise} \end{cases} \quad \phi_-(x) = \begin{cases} -\phi(x) & \text{if } \phi(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

We have that  $\phi_+, \phi_- \in \ell^1(X)$  and  $\phi = \phi_+ - \phi_-$ . If  $\phi \in \ell_c^1(X)$ , then it is also the case that  $\phi_+, \phi_- \in \ell_c^1(X)$ .

It is a standard fact that  $\ell^1(X)$  is a Banach space with the norm

$$\|\phi\| = \sum_{x \in X} |\phi(x)|,$$

see, for example, [20, Chapter III, Example 1.9]. The following lemma and proposition are standard facts for which we could find no adequate reference.

**Lemma 2.4.2.** *The space  $\ell_c^1(X)$  is dense in  $\ell^1(X)$ .*

*Proof.* Let  $\psi \in \ell^1(X)$ . We want to show that for all  $\epsilon > 0$ , there is a  $\phi_\epsilon \in \ell_c^1(X)$  such that  $\|\psi - \phi\| < \epsilon$ . If  $\psi$  has finite support then we can simply take  $\phi = \psi$ , so we now reduce to the case that  $\psi$  has infinite (hence countable by Corollary 0.1.10) support, which we enumerate as a sequence  $(x_i)_{i \in \mathbb{N}}$ . Let  $\epsilon > 0$ . Since  $\psi$  is absolutely summable, there is an  $N \in \mathbb{N}$  such that  $\left| \sum_{i=1}^{\infty} |\psi(x_i)| - \sum_{i=1}^N |\psi(x_i)| \right| < \epsilon$ . Define

$$\phi(x) = \begin{cases} \psi(x) & \text{if } x = x_i \text{ for some } 0 \leq i \leq N \\ 0 & \text{otherwise} \end{cases}.$$

We can now see that

$$\begin{aligned} \|\psi - \phi\| &= \sum_{x \in X} |(\psi - \phi)(x)| \\ &= \sum_{i=1}^N |\psi(x_i) - \phi(x_i)| + \sum_{i=N+1}^{\infty} |\psi(x_i) - 0| \\ &= \sum_{i=1}^N 0 + \sum_{i=N+1}^{\infty} |\psi(x_i) - 0| \\ &= \sum_{i=N+1}^{\infty} |\psi(x_i) - 0| < \epsilon. \end{aligned}$$

□

**Proposition 2.4.3.** *With the above definitions,  $(\ell_c^1(X), \ell_c^1(X)_+, \tau)$  is a radially compact base-norm space and  $(\ell^1(X), \ell^1(X)_+, \tau)$  is a radially compact Banach base-norm space for any set  $X$ .*

*Proof.* The fact that  $\phi = \phi_+ - \phi_-$  implies that  $\ell_c^1(X)$  and  $\ell^1(X)$  are generated by their positive cones.

If  $\ell_c^1(X)$  is not 0, then since  $\ell_c^1(\emptyset) = 0$ , we have that  $X \neq \emptyset$  and so there is some  $y \in X$ . The function  $\delta_y$  defined by

$$\delta_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

is an element of  $\ell_c^1(X)$ . Then

$$\tau(\delta_y) = \sum_{x \in X} \delta_y(x) = 1$$

so  $\tau$  is not the 0 map on  $\ell_c^1(X)$ , and therefore not on  $\ell^1(X)$  either.

The last part to prove is that, with  $B$  being the base,  $\text{absco}(B)$  is radially compact, as radially compact pre-base-norm spaces are base-norm spaces (Proposition 2.2.6). We show this in  $\ell^1(X)$  first by showing that  $\text{absco}(B)$  is equal to the closed unit ball of the usual norm:

$$U = \{\phi \in \ell^1(X) \mid \|\phi\| \leq 1\} = \left\{ \phi \in \ell^1(X) \mid \sum_{x \in X} |\phi(x)| \leq 1 \right\}.$$

We dispose of the trivial case first. If  $\ell^1(X) = 0$ , then  $U = \{0\} = \text{absco}(B)$ . Now we assume that  $\ell^1(X) \neq 0$  and so  $\text{absco}(B) = \text{co}(-B \cup B)$ .

- $\text{absco}(B) \subseteq U$ : If  $\phi \in \text{co}(-B \cup B)$ , then  $\phi = \alpha\phi_+ + (1 - \alpha)(-\phi_-)$  where  $\phi_+, \phi_- \in B$ , and  $\alpha \in [0, 1]$ . Because  $U$ , being a unit ball, is absolutely convex, it suffices to show that  $B \subseteq U$  to show that any expression  $\alpha\phi_+ - (1 - \alpha)\phi_- \in U$ . If  $\phi \in B$ , we have that

$$\begin{aligned} \|\phi\| &= \sum_{x \in X} |\phi(x)| \\ &= \sum_{x \in X} \phi(x) && \phi \in \ell^1(X)_+ \\ &= \tau(\phi) = 1 && \phi \in B, \end{aligned}$$

so  $\phi \in U$ . Therefore  $\text{absco}(B)$  is radially bounded, and so  $\ell^1(X)$  is a pre-base-norm space.

- $U \subseteq \text{absco}(B)$ : Let  $\phi \in U$ . We first define  $|\phi| = \phi_+ + \phi_-$  and observe that  $|\phi|(x) = |\phi(x)|$ . By assumption

$$\begin{aligned} 1 \geq \|\phi\| &= \sum_{x \in X} |\phi(x)| = \sum_{x \in X} |\phi|(x) = \sum_{x \in X} \phi_+(x) + \sum_{x \in X} \phi_-(x) \\ &= \tau(\phi_+) + \tau(\phi_-) \end{aligned}$$

We now have four cases:

- $\tau(\phi_+) = \tau(\phi_-) = 0$ : By the strict positivity of  $\tau$  (Lemma 2.2.2), we have  $\phi_+ = \phi_- = 0$ , so  $\phi = 0$  and  $\phi \in \text{absco}(B)$  because all absolutely convex sets contain zero.
- $\tau(\phi_+) \neq 0$  but  $\tau(\phi_-) = 0$ : Then  $\phi = \phi_+$ , and  $\tau(\phi)$  is invertible, so  $\tau(\phi)^{-1}\phi \in B$ , and  $\tau(\phi) \leq 1$  implies that the absolutely convex combination  $\tau(\phi)(\tau(\phi)^{-1}\phi) = \phi \in \text{absco}(B)$ .
- $\tau(\phi_-) \neq 0$  but  $\tau(\phi_+) = 0$ : This case is similar to the previous one.
- $\tau(\phi_+) \neq 0$  and  $\tau(\phi_-) \neq 0$ : Define  $\phi'_\pm = \tau(\phi_\pm)^{-1}\phi_\pm$  in each case. Then  $\phi'_\pm \in B$  and  $\tau(\phi_+)\phi'_+ - \tau(\phi_-)\phi'_- = \phi$ . The sum of the traces  $\tau(\phi_+) + \tau(\phi_-) \leq 1$ , so this is an absolutely convex combination and shows that  $\phi \in \text{absco}(B)$ .

Radial compactness follows because the closed unit ball intersecting any ray is a closed bounded subset of that ray and hence compact. Any ray in  $\ell_c^1(X)$  is also a ray in  $\ell^1(X)$  so  $\ell_c^1(X)$  has radially compact unit ball too. Finally,  $\ell^1(X)$  is a Banach base-norm space because it is a Banach space in its usual norm and the base norm coincides with the usual norm because the closed unit balls are the same for each.  $\square$

On maps, we define for  $f : X \rightarrow Y$  in **Set**:

$$\ell^1(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x),$$

where  $\phi \in \ell^1(X)$  and  $y \in Y$ . Each of these sums is absolutely convergent since it has as subset of the terms of an absolutely convergent sum, so this is well-defined. We define  $\ell_c^1(f)$  in the same manner, restricting  $\ell^1(f)$  to  $\ell_c^1(X)$ . Since for each  $\phi \in \ell^1(X)$  only finitely many values of  $x$  have  $\phi(x) \neq 0$ , this is also true for  $\ell^1(f)(\phi)$ , so the above definition has the correct type.

**Proposition 2.4.4.** *As defined,  $\ell_c^1$  is a functor  $\mathbf{Set} \rightarrow \mathbf{BNS}$  and  $\ell^1$  is a functor  $\mathbf{Set} \rightarrow \mathbf{BBNS}$ .*

*Proof.* The proof proceeds by showing this for  $\ell^1$  first and deducing that it is so for  $\ell_c^1$  afterwards.

Let  $f : X \rightarrow Y$  be a function. We must first show that  $\ell^1(f)$  is a positive trace-preserving map. For linearity, consider  $\alpha, \beta \in \mathbb{R}$  and  $\phi, \psi \in \ell^1(X)$ , and  $y \in Y$ . Now

$$\begin{aligned} \ell^1(f)(\alpha\phi + \beta\psi)(y) &= \sum_{x \in f^{-1}(y)} (\alpha\phi + \beta\psi)(x) \\ &= \sum_{x \in f^{-1}(y)} (\alpha\phi(x) + \beta\psi(x)) \\ &= \alpha \sum_{x \in f^{-1}(y)} \phi(x) + \beta \sum_{x \in f^{-1}(y)} \psi(x) \\ &= \alpha \ell^1(f)(\phi)(y) + \beta \ell^1(f)(\psi)(y) \\ &= (\alpha \ell^1(f)(\phi) + \beta \ell^1(f)(\psi))(y). \end{aligned}$$

To show positivity, suppose  $\phi \in \ell^1(X)_+$ . Then

$$\ell^1(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x).$$

This is nonnegative because it is the sum of nonnegative numbers, so  $\ell^1(f)(\phi) \in \ell^1(Y)_+$ .

To show that  $\ell^1(f)$  is trace-preserving, let  $\tau$  denote the trace of  $\ell^1(X)$  and  $\sigma$  that of  $\ell^1(Y)$ . We want to show  $\sigma \circ \ell^1(f) = \tau$ . We start with  $\phi \in \ell^1(X)$ :

$$\sigma(\ell^1(f)(\phi)) = \sum_{y \in Y} \ell^1(f)(\phi)(y) = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \phi(x) = \sum_{x \in X} \phi(x) = \tau(\phi).$$

Since  $\ell_c^1(f)$  is the restriction of  $\ell^1(f)$  to  $\ell^1(X)$ , it is also linear, positive and trace preserving and so defines a **BNS** map  $\ell_c^1(X) \rightarrow \ell_c^1(Y)$ .

We must now show that  $\ell^1$  is functorial, *i.e.* that it preserves identity maps and composition. To show the preservation of identity maps, consider  $\text{id}_X : X \rightarrow X$  for an arbitrary set  $X$ , and let  $\phi \in \ell^1(X)$  and  $x \in X$ . Then

$$\ell^1(\text{id}_X)(\phi)(x) = \sum_{x \in \text{id}_X^{-1}(x)} \phi(x) = \phi(x),$$

therefore  $\ell^1(\text{id}_X)(\phi) = \phi$  and so  $\ell^1(\text{id}_X) = \text{id}_{\ell^1(X)}$ .

To show that  $\ell^1$  preserves composition, consider three sets  $X, Y$  and  $Z$ , and two functions  $f : X \rightarrow Y$  and  $Y \rightarrow Z$ , and let  $\phi \in \ell^1(X)$  and  $z \in Z$ . Then

$$\begin{aligned}
 (\ell^1(g) \circ \ell^1(f))(\phi)(z) &= \ell^1(g)(\ell^1(f)(\phi))(z) \\
 &= \sum_{y \in g^{-1}(z)} \ell^1(f)(\phi)(y) \\
 &= \sum_{y \in g^{-1}(z)} \sum_{x \in f^{-1}(y)} \phi(x) \\
 &= \sum_{x \in f^{-1}(g^{-1}(z))} \phi(x) \\
 &= \sum_{x \in (g \circ f)^{-1}(z)} \phi(x) \\
 &= \ell^1(g \circ f)(\phi)(z),
 \end{aligned}$$

applying functional extensionality twice, we get the required functoriality  $\ell^1(g) \circ \ell^1(f) = \ell^1(g \circ f)$ . As  $\ell_c^1$  is defined by restricting  $\ell^1$ ,  $\ell_c^1$  is also a functor.  $\square$

The functor  $\ell_c^1$  can also be composed with the inclusion  $\mathbf{BNS} \hookrightarrow \mathbf{BNS}_{\leq 1}$  to get a functor  $\mathbf{Set} \rightarrow \mathbf{BNS}_{\leq 1}$ , and similarly for  $\ell^1$ , and in fact  $\ell_c^1$  can also be composed with the inclusion  $\mathbf{BNS} \hookrightarrow \mathbf{PreBNS}$ .

The following theorem is the analogue for base-norm spaces of Pumplün and Röhrle's result on the unit ball functor on normed spaces [99].

**Theorem 2.4.5.**  $\ell_c^1 : \mathbf{Set} \rightarrow \mathbf{PreBNS}$  is left adjoint to  $B_{\mathbf{Set}}$  and  $B_{\mathbf{Set}}^{\leq 1}$ , and  $\ell^1 : \mathbf{Set} \rightarrow \mathbf{BBNS}$  is left adjoint to  $B_{\mathbf{Set}}$  and  $B_{\mathbf{Set}}^{\leq 1}$  when restricted to Banach base-norm spaces.

*Proof.* In general we work with  $\ell^1$  as this is the more difficult case, showing how the  $\ell_c^1$  case differs when necessary.

We use a unit and its universal property to define the adjunctions (Theorem 0.4.1 (ii)). We define the units as follows, for  $X$  a set and  $x, x' \in X$ :

$$\begin{aligned}
 \eta_X : X &\rightarrow B_{\mathbf{Set}}(\ell^1(X)) & \eta_X^{\leq 1} : X &\rightarrow B_{\mathbf{Set}}^{\leq 1}(\ell^1(X)) \\
 \eta_X(x)(x') &= \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} & \eta_X^{\leq 1}(x)(x') &= \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

We show that  $\eta_X(x) \in B_{\mathbf{Set}}(\ell_c^1(X))$ . This implies  $\eta_X^{\leq 1}(x) \in B_{\mathbf{Set}}^{\leq 1}(\ell_c^1(X))$  as the definition is the same, and also implies that  $\eta_X(x) \in B_{\mathbf{Set}}(\ell^1(X))$  and



$\eta_X^{\leq 1}(x) \in B_{\mathbf{Set}}^{\leq 1}(\ell^1(X))$ . First, observe that  $\eta_X(x)(x')$  has finite support and is only 1 or 0 so is in  $\ell_c^1(X)_+$ . Secondly, taking the trace

$$\tau(\eta_X(x)) = \sum_{x' \in X} \eta_X(x)(x') = 1,$$

which shows  $\eta_X(x) \in B_{\mathbf{Set}}(\ell_c^1(X))$ .

To show that  $\eta_X$  and  $\eta_X^{\leq 1}$  are natural, we again show only the proof for  $\eta_X$ , as the proof for  $\eta_X^{\leq 1}$  is essentially identical. We want to show that for any function  $f : X \rightarrow Y$ ,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & B_{\mathbf{Set}}(\ell^1(X)) \\ f \downarrow & & \downarrow B_{\mathbf{Set}}(\ell^1(f)) \\ Y & \xrightarrow{\eta_Y} & B_{\mathbf{Set}}(\ell^1(Y)) \end{array}$$

commutes, *i.e.* that  $B_{\mathbf{Set}}(\ell^1(f)) \circ \eta_X = \eta_Y \circ f$ . So let  $x \in X$  and  $y \in Y$ . For the lower left path we have that  $\eta_Y(f(x))(y)$  is 1 if  $f(x) = y$  and 0 otherwise. For the upper right path we have

$$B_{\mathbf{Set}}(\ell^1(f))(\eta_X(x))(y) = \sum_{x' \in f^{-1}(y)} \eta(x)(x').$$

The right hand side is 1 only if  $x \in f^{-1}(y)$ , otherwise it is 0. In other words, it is 1 if  $f(x) = y$  and 0 otherwise. Therefore the two paths are equal by functionality.

We now prove the universal property. In the following, we do the  $\eta_X$  case in full, and the  $\eta_X^{\leq 1}$  case only when it differs (the unique map need only be trace-reducing, not trace preserving). We will also only give the  $\ell^1$  case in full, as the  $\ell_c^1$  case is mostly a restriction of it. We want to show that for every set  $X$  and Banach base-norm space  $(E, E_+, \sigma)$ , given a function  $f : X \rightarrow B_{\mathbf{Set}}(E)$ , there is a unique  $g \in \mathbf{BBNS}(\ell^1(X), E)$  such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & B_{\mathbf{Set}}(\ell^1(X)) \\ & \searrow f & \downarrow B_{\mathbf{Set}}(g) \\ & & B_{\mathbf{Set}}(E). \end{array} \quad (2.1)$$

In the  $\ell_c^1$  case we only assume that  $E$  is a pre-base-norm space in the above, not necessarily a Banach base-norm space.

We define  $g$  as follows (in both cases) for  $\phi \in \ell^1(X)$  as:

$$g(\phi) = \sum_{x \in X} \phi(x)f(x).$$

We first need to show that  $g(\phi)$  defines an element of  $E$ . We have that

$$\|\phi(x)f(x)\| = |\phi(x)|\|f(x)\| \leq \phi(x)$$

because  $f(x) \in B_{\mathbf{Set}}^{\leq 1}$  and  $B_{\mathbf{Set}}^{\leq 1}$  is a subset of the unit ball of  $E$ . So by Lemma A.1.2

$$\sum_{x \in X} \|\phi(x)f(x)\| \leq \sum_{x \in X} |\phi(x)|.$$

Therefore  $(\phi(x)f(x))_{x \in X}$  is an absolutely summable family in  $E$ , a Banach space, so its sum converges by Lemma 0.1.11. In the  $\ell_c^1$  case, the sum is finite and so the previous step is not necessary.

To show  $g$  is linear, let  $\alpha, \beta \in \mathbb{R}$  and  $\phi, \psi \in \ell^1(X)$ . As in the previous part of the proof, we can see that  $\sum_{x \in X} \alpha\phi(x)f(x)$  and  $\sum_{x \in X} \beta\psi(x)f(x)$  are absolutely convergent in  $E$ . We can therefore apply Lemma 0.1.12 to conclude that  $\sum_{x \in X} (\alpha\psi + \beta\phi)(x)f(x) = \sum_{x \in X} \alpha\psi(x)f(x) + \sum_{x \in X} \beta\phi(x)f(x)$ , and that the former sum converges. Then:

$$\begin{aligned} g(\alpha\psi + \beta\phi) &= \sum_{x \in X} (\alpha\psi + \beta\phi)(x)f(x) \\ &= \sum_{x \in X} \alpha\psi(x)f(x) + \sum_{x \in X} \beta\phi(x)f(x) \\ &= \alpha \sum_{x \in X} \psi(x)f(x) + \beta \sum_{x \in X} \phi(x)f(x) \\ &= \alpha g(\psi) + \beta g(\phi). \end{aligned}$$

Now, if  $\phi \in \ell^1(X)_+$ , then

$$g(\phi) = \sum_{x \in X} \phi(x)f(x),$$

and each  $\phi(x)f(x) \in E_+$ , as  $E_+$  is a cone, and the partial sums are in  $E_+$  for the same reason. Since  $E_+$  is closed,  $g(\phi) \in E_+$ , and  $g$  is a positive map. In the  $\ell_c^1$  case, we only assume that  $E$  is a pre-base-norm space, so we do not have that  $E_+$  is closed. However, in this case the sum is finite and so is an element of  $E_+$  simply because it is a cone.

- If  $f : X \rightarrow B_{\mathbf{Set}}(E)$ ,  $g$  is trace-preserving:

We want to show that  $\sigma \circ g = \tau$ , where  $\tau$  is the trace of  $\ell^1(X)$  and  $\sigma$  that of  $E$ . Let  $\phi \in \ell^1(X)$ . We have

$$\begin{aligned}
 \tau(g(\phi)) &= \tau \left( \sum_{x \in X} \phi(x) f(x) \right) \\
 &= \tau \left( \lim_{j \in \mathcal{P}_{\text{fin}}(X)} \sum_{x \in j} \phi(x) f(x) \right) \\
 &= \lim_{j \in \mathcal{P}_{\text{fin}}(X)} \sum_{x \in j} \phi(x) \tau(f(x)) && \tau \text{ continuous and linear} \\
 &= \sum_{x \in X} \phi(x) \tau(f(x)) \\
 &= \sum_{x \in X} \phi(x) = \sigma(\phi).
 \end{aligned}$$

- If  $f : X \rightarrow B_{\mathbf{Set}}^{\leq 1}(E)$ ,  $g$  is trace-reducing:

Let  $\phi \in \ell^1(X)_+$ . We want to show that  $\tau(g(\phi)) \leq \sigma(\phi)$ . So

$$\begin{aligned}
 \tau(g(\phi)) &= \sum_{x \in X} \phi(x) \tau(f(x)) && \text{by previous proof} \\
 &\leq \sum_{x \in X} \phi(x) && \text{since } f(x) \in B_{\mathbf{Set}}^{\leq 1}(E) \\
 &= \sigma(\phi).
 \end{aligned}$$

We now show that  $B_{\mathbf{Set}}(g)$  and  $B_{\mathbf{Set}}^{\leq 1}(g)$  make their respective versions of (2.1) commute and that  $g$  is the unique such map. The proofs of the  $B_{\mathbf{Set}}$  and  $B_{\mathbf{Set}}^{\leq 1}$  cases look identical, so we only give the proof for  $B_{\mathbf{Set}}(g)$ . If  $x \in X$ , then

$$B_{\mathbf{Set}}(g)(\eta_X(x)) = g(\eta_X(x)) = g(\eta_X(x)) = \sum_{x' \in X} \eta_X(x)(x') f(x') = f(x).$$

Finally we show the uniqueness. Suppose  $B_{\mathbf{Set}}(h)$  makes (2.1) commute in place of  $B_{\mathbf{Set}}(g)$ , *i.e.*  $B_{\mathbf{Set}}(h) \circ \eta_X = f$ . If  $\psi \in \ell_c^1(X)$ , we have

$$\psi = \sum_{i=1}^n \psi(x_i) \eta_X(x_i)$$

where the elements  $x_i \in X$  are an enumeration of the support of  $\psi$ . This implies that

$$\begin{aligned}
 h(\psi) &= h\left(\sum_{i=1}^n \psi(x_i)\eta_X(x_i)\right) = \sum_{i=1}^n \psi(x_i)h(\eta_X(x_i)) \\
 &= \sum_{i=1}^n \psi(x_i)B_{\mathbf{Set}}(h)(\eta_X(x_i)) = \sum_{i=1}^n \psi(x_i)f(x_i) \\
 &= \sum_{i=1}^n \psi(x_i)B_{\mathbf{Set}}(g)(\eta_X(x_i)) = \sum_{i=1}^n \psi(x_i)g(\eta_X(x_i)) \\
 &= g\left(\sum_{i=1}^n \psi(x_i)\eta_X(x_i)\right) = g(\psi).
 \end{aligned}$$

We have therefore finished in the  $\ell_c^1$  case. In the  $\ell^1$  case, we use the fact that  $\ell_c^1(X)$  is dense in  $\ell^1(X)$  (Lemma 2.4.2) and that  $g, h$  are continuous maps (Proposition 2.2.12) to deduce  $h = g$ .  $\square$

The existence of these adjunctions has a useful consequence once we have identified the monads arising from them.

**Proposition 2.4.6.** *We have the following identities of monads:*

$$\begin{aligned}
 (B_{\mathbf{Set}}\ell_c^1, \eta, B_{\mathbf{Set}}\epsilon\ell_c^1) &= (\mathcal{D}, \eta, \mu) \\
 (B_{\mathbf{Set}}^{\leq 1}\ell_c^1, \eta, B_{\mathbf{Set}}^{\leq 1}\epsilon\ell_c^1) &= (\mathcal{D}^{\leq 1}, \eta, \mu) \\
 (B_{\mathbf{Set}}\ell^1, \eta, B_{\mathbf{Set}}\epsilon\ell^1) &= (\mathcal{D}_{\infty}, \eta, \mu) \\
 (B_{\mathbf{Set}}^{\leq 1}\ell^1, \eta, B_{\mathbf{Set}}^{\leq 1}\epsilon\ell^1) &= (\mathcal{D}_{\infty}^{\leq 1}, \eta, \mu)
 \end{aligned}$$

*Proof.* That  $B_{\mathbf{Set}}\ell_c^1 = \mathcal{D}$  and the units are equal, and the analogous statements for the other three monads is immediate for the definitions. We therefore only need to prove that  $B_{\mathbf{Set}}\epsilon\ell_c^1 = \mu$  and the analogous statements for the other three monads. The argument is virtually the same in all four cases, so we only show that  $\mathcal{D}_{\infty}$  case. The counit arises from the universal property of  $\eta$  in the following manner, where  $E$  is a Banach base-norm space:

$$\begin{array}{ccc}
 B_{\mathbf{Set}}(E) & \xrightarrow{\eta_X} & B_{\mathbf{Set}}(\ell^1(B_{\mathbf{Set}}(E))) & \ell^1(B_{\mathbf{Set}}(E)) \\
 & \searrow \text{id}_{B_{\mathbf{Set}}(E)} & \downarrow B_{\mathbf{Set}}(\epsilon_E) & \downarrow \epsilon_E \\
 & & B_{\mathbf{Set}}(E) & E.
 \end{array}$$

Therefore, for any  $\phi \in \ell^1(B_{\mathbf{Set}}(E))$ ,

$$\epsilon_E(\phi) = \sum_{x \in B_{\mathbf{Set}}(E)} \phi(x) \cdot x.$$

We can now see that, given  $\Phi \in B_{\mathbf{Set}}(\ell^1(B_{\mathbf{Set}}(\ell^1(X)))) = \mathcal{D}_{\infty}^2(X)$  and  $x \in X$  we have

$$\begin{aligned} B_{\mathbf{Set}}(\epsilon_{\ell^1(X)})(\Phi)(x) &= \epsilon_{\ell^1(X)}(\Phi)(x) \\ &= \left( \sum_{\phi \in B_{\mathbf{Set}}(\ell^1(X))} \Phi(\phi) \cdot \phi \right) (x) \\ &= \sum_{\phi \in \mathcal{D}_{\infty}(X)} \Phi(\phi) \cdot \phi(x) \\ &= \mu_X(\Phi)(x). \end{aligned}$$

□

Thus we have comparison functors

$$B^{\mathcal{D}} : \mathbf{PreBNS} \rightarrow \mathcal{EM}(\mathcal{D}) \quad (2.2)$$

$$B^{\mathcal{D}^{\leq 1}} : \mathbf{PreBNS}_{\leq 1} \rightarrow \mathcal{EM}(\mathcal{D}^{\leq 1}) \quad (2.3)$$

$$B^{\mathcal{D}_{\infty}} : \mathbf{BBNS} \rightarrow \mathcal{EM}(\mathcal{D}_{\infty}) \quad (2.4)$$

$$B^{\mathcal{D}_{\infty}^{\leq 1}} : \mathbf{BBNS}_{\leq 1} \rightarrow \mathcal{EM}(\mathcal{D}_{\infty}^{\leq 1}). \quad (2.5)$$

The monad morphism from  $\mathcal{D} \Rightarrow \mathcal{D}_{\infty}$  induces a functor  $\mathcal{EM}(\mathcal{D}_{\infty}) \rightarrow \mathcal{EM}(\mathcal{D})$  (Proposition 2.4.1), so it seems that we have two functors  $\mathbf{BBNS} \rightarrow \mathcal{EM}(\mathcal{D})$  and  $\mathbf{BBNS}_{\leq 1} \rightarrow \mathcal{EM}(\mathcal{D}^{\leq 1})$ . In fact, they are the same:

**Lemma 2.4.7.** *The following diagram commutes (strictly)*

$$\begin{array}{ccc} \mathbf{BBNS} & \xrightarrow{B^{\mathcal{D}_{\infty}}} & \mathcal{EM}(\mathcal{D}_{\infty}) \\ U \downarrow & & \downarrow V \\ \mathbf{PreBNS} & \xrightarrow{B^{\mathcal{D}}} & \mathcal{EM}(\mathcal{D}), \end{array}$$

where  $U$  is the inclusion functor and  $V$  is the functor arising from the monad morphism  $\mathcal{D} \Rightarrow \mathcal{D}_{\infty}$ . The analogous diagram for  $\mathbf{BBNS}_{\leq 1}$  also commutes.

*Proof.* It is clear that these functors coincide on morphisms, being restriction to the base, or subbase for the  $\mathbf{BBNS}_{\leq 1}$  case. On a Banach base-norm space  $(E, E_+, \tau)$ , the lower left path gives  $(B_E, B_{\mathbf{Set}}(\epsilon_E))$ ,  $\epsilon_E$  being the counit for the adjunction involving  $\ell_c^1$ . The upper right path gives  $(B_E, B_{\mathbf{Set}}(\epsilon_E) \circ \tau_{B_E})$ . In Proposition 2.4.6 we saw that  $\epsilon_E$  for the  $\ell_c^1$  adjunction was the nearly same as the definition for the  $\ell^1$  adjunction, only restricted to elements of finite support. This is exactly what precomposing with  $\tau_{B_E}$  does, so  $B_{\mathbf{Set}}(\epsilon_E) = B_{\mathbf{Set}}(\epsilon_E) \circ \tau_{B_E}$  and the objects are equal as well. The proof for  $\tau^{\leq 1}$  and  $\mathbf{BBNS}_{\leq 1}$  is similar.  $\square$

We therefore can use the name  $B^{\mathcal{D}}$  interchangeably for either functor  $\mathbf{BBNS} \rightarrow \mathcal{EM}(\mathcal{D})$ .

**Proposition 2.4.8.** *The functors  $B^{\mathcal{D}} : \mathbf{PreBNS} \rightarrow \mathcal{EM}(\mathcal{D})$  and  $B^{\mathcal{D}^{\leq 1}} : \mathbf{PreBNS}_{\leq 1} \rightarrow \mathcal{EM}(\mathcal{D}^{\leq 1})$  are full and faithful, and therefore so are  $B^{\mathcal{D}} : \mathbf{BBNS} \rightarrow \mathcal{EM}(\mathcal{D})$  and  $B^{\mathcal{D}^{\leq 1}} : \mathbf{BBNS}_{\leq 1} \rightarrow \mathcal{EM}(\mathcal{D}^{\leq 1})$ .*

*Proof.*

- $B^{\mathcal{D}}$  is faithful:

Let  $f, g : E \rightarrow F$  in  $\mathbf{PreBNS}$ , with  $B^{\mathcal{D}}(f) = B^{\mathcal{D}}(g)$ . If  $E = 0$ , then  $f = g$  already, so we reduce to the case that  $E \neq 0$ . Every element  $x \in E$  can be expressed as  $\alpha x_+ - \beta x_-$  with  $x_+, x_- \in B^{\mathcal{D}}(E)$ . Then

$$\begin{aligned} f(x) &= f(\alpha x_+ - \beta x_-) = \alpha f(x_+) - \beta f(x_-) \\ &= \alpha g(x_+) - \beta g(x_-) = g(\alpha x_+ - \beta x_-) \\ &= g(x). \end{aligned}$$

The proof for  $B^{\mathcal{D}^{\leq 1}}$  is similar.

- $B^{\mathcal{D}}$  is full:

Consider a map  $f : B^{\mathcal{D}}(E) \rightarrow B^{\mathcal{D}}(F)$  in  $\mathcal{EM}(\mathcal{D})$ .

If  $E = 0$ , then  $B^{\mathcal{D}}(E) = \emptyset$  and  $f$  is the unique empty function. Take  $g : E \rightarrow F$  to be the unique map  $0 \rightarrow F$ . Then  $B^{\mathcal{D}}(g) = f$ . We therefore reduce to the case that  $B^{\mathcal{D}}(E) \neq \emptyset$  and so every  $x \in E$  can be expressed as  $\alpha x_+ - \beta x_-$ , with  $x_+, x_- \in B^{\mathcal{D}}(E)$  and  $\alpha, \beta \in [0, \infty)$ . We then attempt to define  $\tilde{f}(\alpha x_+ - \beta x_-) = \alpha f(x_+) - \beta f(x_-)$ , and  $\tilde{f}(0) = 0$  in the case that  $E = 0$ . We first prove this defines a function  $E \rightarrow F$ . Suppose  $\alpha x_+ - \beta x_- = x = \alpha' x'_+ - \beta' x'_-$ . We then have  $\alpha x_+ + \beta' x'_- = \alpha' x'_+ + \beta x_-$ . Taking the trace of both sides, we get  $\alpha + \beta' = \alpha' + \beta$ , and we give the

name  $\gamma$  to this quantity. If  $\gamma = 0$ , we have  $\alpha = \beta = \alpha' = \beta' = 0$ , so  $x = 0$ , and  $0 \cdot f(x_+) - 0 \cdot f(x_-) = 0 = 0 \cdot f(x'_+) - 0 \cdot f(x'_-)$ , so  $\tilde{f}$  is well-defined in this case.

Now we can reduce to the case that  $\gamma > 0$ , so we have the convex combinations  $\frac{\alpha}{\gamma}x_+ + \frac{\beta'}{\gamma}x'_-$  and  $\frac{\alpha'}{\gamma}x'_+ + \frac{\beta}{\gamma}x_-$ . Since  $f$  is an  $\mathcal{EM}(\mathcal{D})$ -morphism, it is affine, so

$$\begin{aligned} \frac{\alpha}{\gamma}f(x_+) + \frac{\beta'}{\gamma}f(x'_-) &= f\left(\frac{\alpha}{\gamma}x_+ + \frac{\beta'}{\gamma}x'_-\right) = f\left(\frac{\alpha'}{\gamma}x'_+ + \frac{\beta}{\gamma}x_-\right) \\ &= \frac{\alpha'}{\gamma}f(x'_+) + \frac{\beta}{\gamma}f(x_-). \end{aligned}$$

Multiplying the equation through by  $\gamma$ , we obtain

$$\alpha f(x_+) + \beta' f(x'_-) = \alpha' f(x'_+) + \beta f(x_-)$$

and we arrive at the conclusion that

$$\begin{aligned} \tilde{f}(\alpha x_+ - \beta x_-) &= \alpha f(x_+) - \beta f(x_-) = \alpha' f(x'_+) - \beta' f(x'_-) \\ &= \tilde{f}(\alpha' x'_+ - \beta' x'_-), \end{aligned}$$

which shows that  $\tilde{f}$  is well-defined.

We now show that  $\tilde{f}$  is linear. Let  $x, y \in E$ , and decompose them as  $x = \alpha x_+ - \beta x_-$  and  $y = \gamma y_+ - \delta y_-$ . The first case is where  $\alpha + \gamma > 0$  and  $\beta + \delta > 0$ . Then we have

$$\begin{aligned} \tilde{f}(x+y) &= \tilde{f}\left((\alpha+\gamma)\left(\frac{\alpha}{\alpha+\gamma}x_+ + \frac{\gamma}{\alpha+\gamma}y_+\right) - (\beta+\delta)\left(\frac{\beta}{\beta+\delta}x_- + \frac{\delta}{\beta+\delta}y_-\right)\right) \\ &= (\alpha+\gamma)f\left(\frac{\alpha}{\alpha+\gamma}x_+ + \frac{\gamma}{\alpha+\gamma}y_+\right) - (\beta+\delta)f\left(\frac{\beta}{\beta+\delta}x_- + \frac{\delta}{\beta+\delta}y_-\right) \\ &= \alpha f(x_+) + \gamma f(y_+) - \beta f(x_-) - \delta f(y_-) \\ &= \alpha f(x_+) - \beta f(x_-) + \gamma f(y_+) - \delta f(y_-) \\ &= \tilde{f}(x) + \tilde{f}(y). \end{aligned}$$

Now, if  $\alpha + \gamma = 0$ , then  $\alpha = \gamma = 0$ . So

$$\begin{aligned}
 \tilde{f}(x+y) &= \tilde{f}(0 \cdot x_+ - \beta x_- + 0 \cdot y_+ - \delta y_-) \\
 &= \tilde{f}\left(0 \cdot \left(\frac{1}{2}x_+ + \frac{1}{2}y_+\right) - (\beta + \delta) \left(\frac{\beta}{\beta + \delta}x_- + \frac{\delta}{\beta + \delta}y_-\right)\right) \\
 &= 0 \cdot f\left(\frac{1}{2}x_+ + \frac{1}{2}y_+\right) - (\beta + \delta)f\left(\frac{\beta}{\beta + \delta}x_- + \frac{\delta}{\beta + \delta}y_-\right) \\
 &= 0 \cdot f(x_+) + 0 \cdot f(y_+) - \beta f(x_-) - \delta f(y_-) \\
 &= 0 \cdot f(x_+) - \beta f(x_-) + 0 \cdot f(y_+) - \delta f(y_-) \\
 &= \tilde{f}(x) + \tilde{f}(y).
 \end{aligned}$$

The case that  $\beta + \delta = 0$  is similar. If both are zero, we have  $\tilde{f}(0) = 0$ , finishing this case. This proves that  $\tilde{f}$  is a homomorphism of abelian groups. We now consider multiplication by a real number  $\xi$ . There are three cases,  $\xi = 0$ ,  $\xi > 0$  and  $\xi < 0$ . We already have  $\xi = 0$ , so we concern ourselves only with the other two cases. If  $\xi > 0$ , we have

$$\begin{aligned}
 \tilde{f}(\xi x) &= \tilde{f}(\xi(\alpha x_+ - \beta x_-)) = \tilde{f}(\xi \alpha x_+ - \xi \beta x_-) = \xi \alpha f(x_+) - \xi \beta f(x_-) \\
 &= \xi(\alpha f(x_+) - \beta f(x_-)) = \xi(\tilde{f}(\alpha x_+ - \beta x_-)) = \xi \tilde{f}(x).
 \end{aligned}$$

If  $\xi < 0$ , we have

$$\begin{aligned}
 \tilde{f}(\xi x) &= \tilde{f}(\xi(\alpha x_+ - \beta x_-)) = \tilde{f}(-\xi \beta x_- - (-\xi \alpha x_+)) \\
 &= -\xi \beta f(x_-) - (-\xi \alpha) f(x_+) = \xi \alpha f(x_+) - \xi \beta f(x_-) \\
 &= \xi(\alpha f(x_+) - \beta f(x_-)) = \xi(\tilde{f}(\alpha x_+ - \beta x_-)) \\
 &= \xi \tilde{f}(x).
 \end{aligned}$$

This completes the proof of linearity. If  $x \in E_+$ , then we can express it as  $\alpha x_+$  for  $x_+ \in B^{\mathcal{D}}(E)$ , by dividing by its trace or taking  $\alpha = 0$  if  $x = 0$ . We can therefore write it as  $\alpha x_+ - \beta x_-$  with  $\beta = 0$ , taking  $x_- = x_+$  if necessary. We have

$$\tilde{f}(x) = \alpha f(x_+) - \beta f(x_-) = \alpha f(x_+),$$

and then because  $\alpha \geq 0$  and  $f(x_+) \in B^{\mathcal{D}}(F)$  and  $F_+$  is a cone, we have that  $\alpha f(x_+) \in F_+$ , establishing the positivity of  $\tilde{f}$ .

For trace-preservation, we observe first that if  $E = 0$ , the trace  $\tau(0) = 0$  and  $\sigma(\tilde{f}(0)) = \sigma(0) = 0$ , so the trace is preserved. We therefore reduce



to the case that  $E \neq 0$ . For  $\tau$  we have

$$\tau(x) = \tau(\alpha x_+ - \beta x_-) = \alpha \tau(x_+) - \beta \tau(x_-) = \alpha - \beta.$$

For  $\sigma \circ \tilde{f}$  we have

$$\begin{aligned} \sigma(\tilde{f}(x)) &= \sigma(\tilde{f}(\alpha x_+ - \beta x_-)) = \sigma(\alpha f(x_+) - \beta f(x_-)) \\ &= \alpha \sigma(f(x_+)) - \beta \sigma(f(x_-)) = \alpha - \beta. \end{aligned}$$

Since the two are equal, we have trace-preservation as well, and  $\tilde{f}$  is a **PreBNS** morphism such that  $B^{\mathcal{D}}(\tilde{f}) = f$ , as required.

- $B^{\mathcal{D}^{\leq 1}}$  is full: We use the same definition for  $\tilde{f}$  as in the  $B^{\mathcal{D}}$  case, and then the proofs of well-definedness, linearity and positivity are virtually the same. We therefore only present the proof that  $\tilde{f}$  is trace-reducing.

In the case that  $B^{\mathcal{D}^{\leq 1}}(E) = \{0\}$ , then  $E = 0$  and

$$(\tau - \sigma \circ \tilde{f})(0) = \tau(0) - \sigma(\tilde{f}(0)) = 0 - 0 = 0$$

so  $\sigma \circ \tilde{f} \leq \tau$ . We therefore reduce to the case that  $E \neq 0$ . If  $x \in E_{+i}$  then  $x = \alpha x_+ - \beta x_-$  with  $\beta = 0$ , and  $x_+, x_- \in B^{\mathcal{D}}(E)$ . We have that  $\tau(x) = \tau(\alpha x_+) = \alpha$ . Then

$$\begin{aligned} \sigma(\tilde{f}(x)) &= \sigma(\tilde{f}(\alpha x_+ - \beta x_-)) = \sigma(\alpha f(x_+) - \beta f(x_-)) \\ &= \alpha \sigma(f(x_+)) - 0 \leq \alpha = \tau(x), \end{aligned}$$

which shows that  $\tilde{f}$  is trace-reducing.  $\square$

The preceding proof also shows the following:

**Corollary 2.4.9.** *The functor  $B : \mathbf{PreBNS} \rightarrow \mathbf{BConv}$  is an equivalence.*

*Proof.* It is full and faithful by the previous proposition, as affine maps and  $\mathcal{EM}(\mathcal{D})$  maps coincide for convex sets. We have already shown that it is essentially surjective in Proposition 2.2.13, so it is definition (iii) of an equivalence (Theorem 0.4.3).  $\square$

By combining this corollary with Lemma 2.4.7, we see that if  $X$  and  $Y$  are  $\mathcal{D}_{\infty}$ -algebras isomorphic to bases of pre-base-norm spaces,  $\mathcal{EM}(\mathcal{D})(X, Y) = \mathcal{EM}(\mathcal{D}_{\infty})(X, Y)$ . In fact, we can improve this result somewhat, also extending a result from [98, Theorem 3.6 (i)] (we do not require that the domain be the base of a base-norm space).

**Lemma 2.4.10.** *Let  $(X, \alpha_X)$  and  $(Y, \alpha_Y)$  be Eilenberg-Moore algebras of  $\mathcal{D}_\infty$ , where  $Y \cong B_E$  for some pre-base-norm space  $(E, E_+, \tau)$ . Then the hom-set  $\mathcal{EM}(\mathcal{D})(X, Y) = \mathcal{EM}(\mathcal{D}_\infty)(X, Y)$ . The analogous result also holds for  $\mathcal{D}^{\leq 1}$  and  $\mathcal{D}_\infty^{\leq 1}$ .*

*Proof.* As the inclusion map  $\mathcal{EM}(\mathcal{D}_\infty)(X, Y) \subseteq \mathcal{EM}(\mathcal{D})(X, Y)$  is natural, we can reduce to the case that  $Y = B_E$ . All we need to show is that if  $a : X \rightarrow Y$  is a map of  $\mathcal{D}$ -algebras, it is a map of  $\mathcal{D}_\infty$ -algebras, i.e. that

$$\begin{array}{ccc} \mathcal{D}_\infty(X) & \xrightarrow{\mathcal{D}_\infty(a)} & \mathcal{D}_\infty(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ X & \xrightarrow{a} & Y \end{array}$$

commutes. By Proposition 2.4.8,  $a \circ \alpha_X$  and  $\alpha_Y \circ \mathcal{D}_\infty(a)$  extend to trace-preserving maps  $b, c : \ell^1(X) \rightarrow E$ , which agree on  $\ell_c^1(X)$  because  $a$  is a map of  $\mathcal{D}$ -algebras. By Proposition 2.2.12,  $b$  and  $c$  are bounded, and therefore continuous, and by Lemma 2.4.2 the set  $\ell_c^1(X)$ , on which they agree, is dense,  $b = c$ . Therefore  $a \circ \alpha_X = \alpha_Y \circ \mathcal{D}_\infty(a)$ , as required.

The proof for  $\mathcal{D}^{\leq 1}$  and  $\mathcal{D}_\infty^{\leq 1}$  is similar, using the  $\mathcal{D}^{\leq 1}$  and  $\mathcal{D}_\infty^{\leq 1}$  parts of the previously mentioned results.  $\square$

At this point, the reader might wonder if there is any difference between  $\mathcal{D}$ -maps and  $\mathcal{D}_\infty$ -maps at all, so we sketch a counterexample. We can define  $(2, \alpha_2)$  as  $2 = \{0, \infty\}$ ,  $\alpha_2(\eta_2(0)) = 0$  and  $\alpha_2(\phi) = \infty$  otherwise, and a map  $f : \mathcal{D}_\infty(\mathbb{N}) \rightarrow 2$  such that  $f(\phi) = 0$  if  $\phi$  has finite support and  $\infty$  if it has infinite support. It is left as an exercise to the reader to show that  $2$  is an Eilenberg-Moore algebra of  $\mathcal{D}_\infty$  and  $f$  a map in  $\mathcal{EM}(\mathcal{D})$ . If we define  $\Phi \in \mathcal{D}_\infty^2(\mathbb{N})$  as  $\Phi(\eta_{\mathbb{N}}(i)) = 2^{-i}$ , we have  $\alpha_2(\mathcal{D}_\infty(f)(\Phi)) = 0$  because each  $\eta_{\mathbb{N}}(i)$  has finite support, but  $f(\mu_{\mathbb{N}}(\Phi)) = \infty$ . This shows that the forgetful functor  $V : \mathcal{EM}(\mathcal{D}_\infty) \rightarrow \mathcal{EM}(\mathcal{D})$  is not full.

The following was first proven in [32, §2], and is also proven in [98, Lemma 3.2] in the setting of superconvex sets. Both proofs are essentially the same, using variants of the argument that we already used in Lemma 2.2.15.

**Proposition 2.4.11.** *Let  $(E, E_+, \tau)$  be a pre-base-norm space. If  $B_E$ , or equivalently  $B_E^{\leq 1}$ , is  $\sigma$ -convex, then  $E$  is a Banach space in the base norm.*

*Proof.* We first show that  $B_E$  is  $\sigma$ -convex iff  $B_E^{\leq 1}$  is. If  $B_E^{\leq 1}$  is  $\sigma$ -convex, then any  $\sigma$ -convex combination  $\sum_{i=1}^\infty \alpha_i x_i$  where  $x_i \in B_E$  is also a  $\sigma$ -convex

combination in  $B_E^{\leq 1}$ , so the sum converges to some  $x \in B_E^{\leq 1}$ . By Lemma 2.2.4

$$\tau \left( \sum_{i=1}^{\infty} \alpha_i x_i \right) = \sum_{i=1}^{\infty} \tau(\alpha_i x_i) = \sum_{i=1}^{\infty} \alpha_i = 1,$$

so  $x \in B_E$ , so  $B_E$  is  $\sigma$ -convex.

In the other direction, suppose  $B_E$  is  $\sigma$ -convex, and let  $\sum_{i=1}^{\infty} \alpha_i x_i$  be a  $\sigma$ -convex combination in  $B_E^{\leq 1}$ . If  $x_i = 0$  for all  $i$  or all but finitely many  $i$ , then  $\sum_{i=1}^{\infty} \alpha_i x_i$  is actually a finite convex combination, and we are done. Therefore we reduce to the case that  $x_i \neq 0$  for infinitely many  $i$ . Define  $(y_i)$  to be the subsequence of non-zero terms, and  $(\beta_i)$  to be the corresponding elements in  $(\alpha_i)$ , and we have that  $\sum_{i=1}^{\infty} \beta_i y_i$  converges iff  $\sum_{i=1}^{\infty} \alpha_i x_i$  does and if they do they have the same limit. Then define  $z_i = \frac{y_i}{\tau(y_i)}$ , which avoids dividing by zero by Lemma 2.2.2, and gives a sequence in  $B_E$ . As  $\sum_{i=1}^{\infty} \alpha_i$  converges, we have that  $\sum_{i=1}^{\infty} \beta_i$  converges absolutely, though perhaps to some value less than 1. We define  $\beta = \sum_{i=1}^{\infty} \beta_i$ , and  $\gamma_i = \frac{\beta_i \tau(y_i)}{\beta}$ . As the sequence  $\frac{\tau(y_i)}{\beta}$  is bounded, we have

$$\sum_{i=1}^{\infty} \gamma_i = \sum_{i=1}^{\infty} \beta_i \cdot \frac{\tau(y_i)}{\beta}$$

converges, and by continuity of scalar multiplication and the definition of  $\beta$  its value is 1. Therefore  $\sum_{i=1}^{\infty} \gamma_i z_i$  is a  $\sigma$ -convex combination in  $B_E$ , and we name the limit of it  $z$ . We then have

$$\beta z = \beta \sum_{i=1}^{\infty} \gamma_i z_i = \sum_{i=1}^{\infty} \beta \cdot \frac{\beta_i \tau(y_i)}{\beta} \cdot \frac{y_i}{\tau(y_i)} = \sum_{i=1}^{\infty} \beta_i y_i.$$

Therefore  $\sum_{i=1}^{\infty} \alpha_i x_i$  converges to  $\beta z$ , so  $B_E^{\leq 1}$  is  $\sigma$ -convex.

By Corollary 2.2.5,  $\text{Ball}(\|\cdot\|) \cap E_+ = B_E^{\leq 1}$ , so if  $B_E^{\leq 1}$  is  $\sigma$ -convex, we may apply Lemma 2.2.15 to conclude that  $E = E_+ - E_+$  is complete in  $\|\cdot\|_{\text{co}(-B^{\leq 1} \cup B^{\leq 1})}$ , which is the base norm by Lemma 2.2.11.  $\square$

The following lemma is original.

**Lemma 2.4.12.**

- (i) Let  $(X, \alpha_X)$  be a  $\mathcal{D}$ -algebra isomorphic to  $B_E$  for some pre-base-norm space  $(E, E_+, \tau)$ . If  $\beta_X, \gamma_X : \mathcal{D}_{\infty}(X) \rightarrow X$  are  $\mathcal{D}_{\infty}$ -algebra structures agreeing on  $\mathcal{D}(X)$  with  $\alpha_X$ , then  $\beta_X = \gamma_X$ , and  $B_E$  is  $\sigma$ -convex for the base norm.

- (ii) If  $(E, X)$  is an object of  $\mathbf{BConv}$ ,  $\mathcal{T}$  is the topology on  $E$ , and  $X$  is  $\sigma$ -convex in  $\mathcal{T}$ , then  $\sigma$ -convex combinations extend the  $\mathcal{D}$ -algebra structure of  $X$  to a  $\mathcal{D}_\infty$ -algebra structure, so  $X$  is  $\sigma$ -convex in the base norm of the base-norm space constructed in Proposition 2.2.13.

*Proof.*

- (i) Let  $i : (X, \alpha_X) \rightarrow B_E$  be an  $\mathcal{EM}(\mathcal{D})$  isomorphism, and suppose that  $\beta_X, \gamma_X : \mathcal{D}_\infty(X) \rightarrow X$  are  $\mathcal{EM}(\mathcal{D}_\infty)$ -structures on  $X$  extending  $\alpha_X$ . Then  $\beta_X$  and  $\gamma_X$  are  $\mathcal{EM}(\mathcal{D}_\infty)$ -maps from  $\mathcal{D}_\infty(X) \rightarrow X$  by the definition of an Eilenberg-Moore algebra. Therefore they are also  $\mathcal{EM}(\mathcal{D})$ -maps (Proposition 2.4.1), and so  $i \circ \beta_X$  and  $i \circ \gamma_X$  are  $\mathcal{EM}(\mathcal{D})$ -maps between bases of pre-base-norm spaces, and therefore extend to trace-preserving maps  $f, g : \ell^1(X) \rightarrow E$  by Proposition 2.4.8. The maps  $f, g$  agree when restricted to  $\mathcal{D}(X)$ , so agree on  $\ell_c^1(X)$  by linearity. As they are continuous (Proposition 2.2.12) and  $\ell_c^1(X)$  is dense (Lemma 2.4.2),  $f = g$ . Therefore  $i \circ \beta_X = i \circ \gamma_X$ , so  $\beta_X = \gamma_X$ .

We now show that  $B_E$  is  $\sigma$ -convex. Let  $\sum_{i=1}^\infty \alpha_i y_i$  be a  $\sigma$ -convex combination of elements of  $B_E$ . Without loss of generality, take  $y_i \neq y_j$  for  $i \neq j$  and  $\alpha_i \neq 0$  for all  $i$ . Let  $x_j = i^{-1}(y_j)$ , which is unique because  $i$  is a bijection. Define  $\phi \in \mathcal{D}_\infty(X)$  as

$$\begin{aligned} \phi(x_i) &= \alpha_i & \text{if } i \in \mathbb{N} \\ \phi(x) &= 0 & \text{otherwise,} \end{aligned}$$

which is in  $\mathcal{D}_\infty(X)$  because  $\alpha_i$  is the coefficients of an absolutely convex combination, and define  $\phi_n \in \ell_c^1(X)$  as

$$\begin{aligned} \phi_n(x_i) &= \alpha_i & \text{if } 1 \leq i \leq n \\ \phi_n(x) &= 0 & \text{otherwise.} \end{aligned}$$

We have

$$\|\phi - \phi_n\| = \sum_{i=n+1}^\infty |\phi(x_i)| = \sum_{i=n+1}^\infty \alpha_i$$

so  $\phi_n \rightarrow \phi$  in the  $\ell^1$  norm. We know

$$f|_{\mathcal{D}(X)} = i \circ \alpha_X = B(\epsilon_E) \circ \mathcal{D}(i)$$

because  $i$  is an  $\mathcal{EM}(\mathcal{D})$ -morphism and the Eilenberg-Moore structure on  $B_E$  comes from a comparison functor. We can define  $\beta_n = \sum_{i=1}^n \alpha_i$  and

$\psi_n = \frac{\phi_n}{\beta_n}$ , it is then an element of  $\mathcal{D}(X)$ , so

$$f(\psi_n) = B(E)(\mathcal{D}(i)(\psi_n)) = \epsilon_E(\psi_n \circ i^{-1}).$$

By the definition of  $\epsilon_E$  from Proposition 2.4.6, this is

$$\sum_{y \in B_E} (\psi_n \circ i^{-1})(y) \cdot y = \sum_{j=1}^n \psi_n(x_j) \cdot y_j = \frac{\sum_{j=1}^n \phi_n(x_j) y_j}{\beta_n} = \frac{\sum_{j=1}^n \alpha_j y_j}{\beta_n}.$$

By linearity of  $f$ , we can cancel the  $\beta_n$  and get  $f(\phi_n) = \sum_{j=1}^n \alpha_j y_j$ . As  $f$  is continuous (Proposition 2.2.12),  $f(\phi_n) \rightarrow f(\phi)$ , so

$$\sum_{j=1}^n \alpha_j y_j \rightarrow f(\phi),$$

so  $\sum_{j=1}^{\infty} \alpha_j y_j$  converges to an element of  $B_E$ . Therefore  $B_E$  is  $\sigma$ -convex in the base norm.

- (ii) We define a  $\mathcal{D}_{\infty}$ -algebra structure  $\beta_X : \mathcal{D}_{\infty}(X) \rightarrow X$  as follows, where  $\phi \in \mathcal{D}_{\infty}(X)$ :

$$\beta_X(\phi) = \sum_{x \in X} \phi(x) \cdot x.$$

By Lemma 0.1.9, the sum defining  $\beta_X(\phi)$  is a  $\sigma$ -convex combination, so defines an element of  $X$ . This definition extends the  $\mathcal{D}$ -algebra structure  $\alpha_X$  defined by Propositions 2.2.13 and 2.4.6. We still need to show that it makes  $(X, \beta_X)$  an Eilenberg-Moore algebra. We have  $\beta_X \circ \eta_X = \text{id}_X$  because the range of  $\eta_X$  lies inside  $\mathcal{D}(X)$ , and we already have  $\alpha_X \circ \eta_X = \text{id}_X$ . Therefore we need to show

$$\begin{array}{ccc} \mathcal{D}_{\infty}^2(X) & \xrightarrow{\mathcal{D}_{\infty}(\beta_X)} & \mathcal{D}_{\infty}(X) \\ \mu_X \downarrow & & \downarrow \beta_X \\ \mathcal{D}_{\infty}(X) & \xrightarrow{\beta_X} & X. \end{array}$$

Let  $\Phi \in \mathcal{D}_\infty^2(X)$ . The lower right path gives

$$\begin{aligned}\beta_X(\mu_X(\Phi)) &= \sum_{x \in X} \mu_X(\Phi)(x) \cdot x \\ &= \sum_{x \in X} \left( \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \right) \cdot x \\ &= \sum_{x \in X} \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \cdot x.\end{aligned}$$

The upper right path gives

$$\begin{aligned}\beta_X(\mathcal{D}_\infty(\beta_X)(\Phi)) &= \sum_{x \in X} \mathcal{D}_\infty(\beta_X)(\Phi)(x) \cdot x \\ &= \sum_{x \in X} \left( \sum_{\phi \in \beta_X^{-1}(x)} \Phi(\phi) \right) \cdot x \\ &= \sum_{x \in X} \sum_{\phi \in \beta_X^{-1}(x)} \Phi(\phi) \cdot x.\end{aligned}$$

For each  $\phi \in \mathcal{D}_\infty(X)$ , there is a unique  $x$  such that  $\beta_X(\phi) = x$ , and all values of  $x$  occur, so we can rewrite the above expression as

$$\begin{aligned}\sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \beta_X(\phi) &= \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \left( \sum_{x \in X} \phi(x) \cdot x \right) \\ &= \sum_{\phi \in \mathcal{D}_\infty(X)} \sum_{x \in X} \Phi(\phi) \cdot \phi(x) \cdot x \\ &= \sum_{x \in X} \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \cdot x,\end{aligned}$$

by the absolute convergence of the sums. Therefore the diagram commutes.

We can then apply the previous part to conclude that the base of the corresponding base-norm space is  $\sigma$ -convex in the base norm.  $\square$

We define **CBCnv** to be the full subcategory of **BConv** on objects  $(E, X)$  where  $X$  is sequentially complete in the subspace uniformity of  $E$ . If  $(E, E_+, \tau)$  is a Banach base-norm space, then  $B_E$  is a closed subspace of the complete

space  $E$ , so  $(E, B_E)$  is an object of **CBConv**, and the functor  $B : \mathbf{PreBNS} \rightarrow \mathbf{BConv}$  restricts to  $B : \mathbf{BBNS} \rightarrow \mathbf{CBConv}$ .

**Proposition 2.4.13.** *The functor  $B : \mathbf{BBNS} \rightarrow \mathbf{CBConv}$  is an equivalence of categories.*

*Proof.* By Proposition 2.4.8 we have that  $B$  is full and faithful because  $\mathcal{EM}(\mathcal{D})$  maps are the same as **CBConv** maps. By Proposition 2.2.13, we have that for any object  $(E, X)$  in **CBConv**, where  $\mathcal{T}$  is the topology of  $E$ , there is a pre-base-norm space  $(F, F_+, \tau)$ , with a locally convex topology  $\mathcal{S}$  on  $F$  such that  $(E, X) \cong (F, B_F)$  in **BConv** and this isomorphism is a uniform isomorphism with respect to the uniformities on  $X$  and  $B_F$  induced by  $\mathcal{T}$  and  $\mathcal{S}$  respectively. Since  $X$  is sequentially complete, by 0.1.19 it is  $\sigma$ -convex, so by Lemma 2.4.12  $B_F$  is  $\sigma$ -convex in the base norm. By Proposition 2.4.11,  $F$  is a Banach space in the base norm. All that is left to show  $(F, F_+, \tau)$  is a Banach base-norm space is to show that  $F_+$  is closed in the base norm. As  $X$  is sequentially complete in  $\mathcal{T}$ , we have that  $B_F$  is sequentially complete in  $\mathcal{S}$ , and therefore sequentially closed. By Lemma 2.2.14,  $F_+$  is therefore sequentially closed in  $(F, \mathcal{S})$ . As the base-norm topology is finer than  $\mathcal{S}$ , we have that  $F_+$  is sequentially closed, and therefore closed, in the base norm.  $\square$

The preceding proposition can be considered to be an extension of [47, Theorem 3.6], replacing completeness of a convex set  $X$  in the intrinsic metric  $\rho$ , defined in that article, by sequential completeness for some locally convex space that contains  $X$ .

If we define **CBConvBan** to be closed (and therefore complete) bounded convex subsets of Banach spaces, the forgetful functor from **CBConvBan**  $\rightarrow$  **CBConv** is an equivalence by the previous proposition, because the image of **BBNS** under  $B$  lies inside **CBConvBan**. In other words, every sequentially complete bounded convex subset of a locally convex space is embeddable as a closed subset of a Banach space. It is not the case that every object of **CCL** can be embedded in a Banach space (with the norm topology, at least) because some objects, such as  $[0, 1]^X$  for uncountable  $X$ , are not first-countable, and therefore not metrizable.

### 2.4.2 Bounded Affine Functions and the Dual Space

If  $(X, \alpha_X)$  is an Eilenberg-Moore algebra of  $\mathcal{D}$ , we can define the real-valued bounded affine functions  $\text{BAff}(X, \alpha_X)$ , and for an Eilenberg-Moore algebra of  $\mathcal{D}_\infty$ , we can define the bounded  $\sigma$ -affine functions  $\text{BAff}_\infty(X, \alpha_X)$ , where we

require that infinite convex combinations are preserved. In more detail

$$\begin{aligned} \text{BAff}(X, \alpha_X) &= \left\{ a \in \ell^\infty(X) \left| \forall \phi \in \mathcal{D}(X). a(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot a(x) \right. \right\} \\ \text{BAff}_\infty(X, \alpha_X) &= \left\{ a \in \ell^\infty(X) \left| \forall \phi \in \mathcal{D}_\infty(X). a(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot a(x) \right. \right\}, \end{aligned}$$

where  $\ell^\infty(X)$  is understood to be the space of bounded real-valued functions. Note that the sum in the definition of  $\text{BAff}_\infty(X, \alpha_X)$  will always converge by Lemmas 0.1.9 and 0.1.19. We will sometimes use  $X$  to refer to  $(X, \alpha_X)$  when there is no possibility of confusion, and therefore  $\text{BAff}(X, \alpha)$  will sometimes be written as  $\text{BAff}(X)$ . We will soon see that  $\text{BAff}_\infty$  and  $\text{BAff}$  are actually the same. To do this, we need a definition. For each  $\alpha \in \mathbb{R}_{>0}$  we can define  $[-\alpha, \alpha] \subseteq \mathbb{R}$ . As it is a complete subset of  $\mathbb{R}$ ,  $(\mathbb{R}, [-\alpha, \alpha])$  is an object of **CBCConv**, so it admits a  $\mathcal{D}_\infty$ -algebra structure. We can construct a  $\mathcal{D}$ -algebra isomorphism  $i_\alpha : [-\alpha, \alpha] \rightarrow B^{\mathcal{D}_\infty}(\mathbb{R}^2)$  as follows

$$i_\alpha(x) = \left( \frac{\alpha - x}{2\alpha}, \frac{\alpha + x}{2\alpha} \right). \quad (2.6)$$

This map is a  $\mathcal{D}$ -algebra homomorphism because it is affine, and it is therefore a  $\mathcal{D}_\infty$ -algebra homomorphism by Lemma 2.4.10. As it is an isomorphism, it is also a  $\mathcal{D}_\infty$ -algebra isomorphism.

**Lemma 2.4.14.** *For all  $\mathcal{D}_\infty$ -algebras  $(X, \alpha_X)$ ,  $\text{BAff}_\infty(X) = \text{BAff}(X)$ .*

*Proof.* By definition,  $\text{BAff}_\infty(X) \subseteq \text{BAff}(X)$ . For the opposite inclusion, let  $a \in \text{BAff}(X)$ , and then there is an  $\alpha \in \mathbb{R}_{>0}$  such that  $a(X) \subseteq [-\alpha, \alpha]$ . So  $a$  can be considered to be a  $\mathcal{D}$ -algebra morphism  $a : X \rightarrow [-\alpha, \alpha]$ . Therefore  $i_\alpha \circ a : X \rightarrow B(\mathbb{R}^2)$  is a  $\mathcal{D}$ -algebra map, so by Lemma 2.4.10 it is an  $\mathcal{EM}(\mathcal{D}_\infty)$ -algebra map. Therefore  $a = i^{-1} \circ i \circ a$  is a  $\mathcal{D}_\infty$ -algebra homomorphism, and therefore an element of  $\text{BAff}_\infty(X)$  as a map  $X \rightarrow \mathbb{R}$ .  $\square$

We can give  $\text{BAff}(X, \alpha)$  the order-unit space structure it should have as a subspace of  $\ell^\infty(X)$ , *i.e.* the vector space operations are pointwise, define  $a : X \rightarrow \mathbb{R}$  to be positive if  $a(x) \geq 0$  for all  $x \in X$ , and define the unit to be the map such that  $u(x) = 1$  for all  $x \in X$ . The following proposition was first proved in [91, §3] in the more general setting where  $X$  is a convex prestructure, rather than a  $\mathcal{D}$  or  $\mathcal{D}_\infty$ -algebra. We give it explicitly here, as a basis for other versions we explore later.



**Proposition 2.4.15.** *The preceding definitions make  $\text{BAff}(X, \alpha)$  into a Banach order-unit space.*

*Proof.*

- $\text{BAff}(X)$  is a subspace of  $\ell^\infty$ : We first show closure under addition. Let  $a, b \in \text{BAff}(X)$ . We know that  $a + b \in \ell^\infty(X)$ , so we only want to show that for all  $\phi \in \mathcal{D}(X)$  that

$$(a + b)(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot (a + b)(x).$$

So let  $\phi \in \mathcal{D}(X)$ . Then

$$\begin{aligned} (a + b)(\alpha_X(\phi)) &= a(\alpha_X(\phi)) + b(\alpha_X(\phi)) = \sum_{x \in X} \phi(x)a(x) + \sum_{x \in X} \phi(x)b(x) \\ &= \sum_{x \in X} (\phi(x)a(x) + \phi(x)b(x)). \end{aligned}$$

The last step is because the sum is finite. We then have that this is equal to

$$= \sum_{x \in X} \phi(x) \cdot (a + b)(x),$$

as required.

To show closure under multiplication, let  $\beta \in \mathbb{R}$  and  $a \in \text{BAff}(X)$ . Then  $\beta a \in \ell^\infty(X)$ , and given  $\phi \in \mathcal{D}(X)$

$$(\beta a)(\alpha_X(\phi)) = \beta a(\alpha_X(\phi)) = \beta \left( \sum_{x \in X} \phi(x) \cdot a(x) \right) = \sum_{x \in X} \phi(x) \cdot (\beta a)(x).$$

- The positive cone is a cone: This follows directly from the fact that it is the restriction of  $\ell^\infty(X)$ 's positive cone.
- The unit is an element of  $\text{BAff}(X)$ :

We already know it is bounded, so we only need to show it is affine. So let  $\phi \in \mathcal{D}(X)$ :

$$u(\alpha_X(\phi)) = 1 = \sum_{x \in X} \phi(x) = \sum_{x \in X} \phi(x) \cdot u(x).$$

- 1 is a strong archimedean unit: This follows directly from it being a strong archimedean unit in  $\ell^\infty(X)$ .
- $\text{BAff}(X)$  is complete in the norm defined by  $[-u, u]$ :

As they are subspaces of  $\ell^\infty(X)$  and  $\ell^\infty(X)$ 's norm is defined by  $[-u, u]$ , it suffices to show that  $\text{BAff}(X)$  is a closed subspace of  $\ell^\infty(X)$ . Let  $(a_i)_{i \in \mathbb{N}}$  be a sequence in  $\text{BAff}(X)$  uniformly converging to  $a \in \ell^\infty(X)$ . If  $\phi \in \mathcal{D}(X)$  then

$$\begin{aligned} a(\alpha_X(\phi)) &= \lim_{i \rightarrow \infty} a_i(\alpha_X(\phi)) = \lim_{i \rightarrow \infty} \sum_{x \in X} \phi(x) \cdot a_i(x) \\ &= \sum_{x \in X} \phi(x) \left( \lim_{i \rightarrow \infty} a_i \right)(x) = \sum_{x \in X} \phi(x) a(x). \end{aligned}$$

so  $a$  is in  $\text{BAff}(X)$ . □

If  $f : (X, \alpha_X) \rightarrow (Y, \alpha_Y)$  is an  $\mathcal{EM}(\mathcal{D})$  morphism, we can define

$$\begin{aligned} \text{BAff}(f) : \text{BAff}(Y) &\rightarrow \text{BAff}(X) \\ \text{BAff}(f)(b) &= b \circ f, \end{aligned}$$

where  $b \in \text{BAff}(Y)$ .

**Theorem 2.4.16.** *These definitions make a functor*

$$\text{BAff} : \mathcal{EM}(\mathcal{D}) \rightarrow \mathbf{BOUS}^{\text{op}}$$

*Proof.* Let  $f : (X, \alpha_X) \rightarrow (Y, \alpha_Y)$  be an  $\mathcal{EM}(\mathcal{D})$  morphism, and  $b$  an element of  $\text{BAff}(Y)$ . First we show that  $b \circ f \in \text{BAff}(X)$ . We know that  $b$  is bounded, so there must exist an  $\alpha \in \mathbb{R}_{\geq 0}$  such that  $\forall y \in Y. |b(y)| \leq \alpha$ . Since for all  $x \in X$ ,  $f(x) \in Y$ , we have that  $\forall x \in X. |b(f(x))| \leq \alpha$ , so  $b \circ f$  is bounded.

For the affineness, let  $\phi \in \mathcal{D}(X)$ . We want to show that

$$\text{BAff}(f)(b)(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot \text{BAff}(f)(b)(x). \quad (2.7)$$

Starting with the left hand side, we have

$$\begin{aligned}
 \text{BAff}(f)(b)(\alpha_X(\phi)) &= b(f(\alpha_X(\phi))) \\
 &= b(\alpha_Y(\mathcal{D}(f)(\phi))) && f \text{ an } \mathcal{EM}(\mathcal{D}) \text{ map} \\
 &= \sum_{y \in Y} \mathcal{D}(f)(\phi)(y) \cdot b(y) && b \in \text{BAff}(Y) \\
 &= \sum_{y \in Y} b(y) \cdot \left( \sum_{x \in f^{-1}(y)} \phi(x) \right),
 \end{aligned}$$

by the definition of  $\mathcal{D}(f)$ . Now, if we look at the right hand side of (2.7), we get

$$\begin{aligned}
 \sum_{x \in X} \phi(x) \cdot \text{BAff}(f)(b)(x) &= \sum_{x \in X} \phi(x) \cdot b(f(x)) \\
 &= \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \phi(x) \cdot b(f(x)) && \text{finite sum} \\
 &= \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \phi(x) \cdot b(y) && f(x) = y \\
 &= \sum_{y \in Y} b(y) \left( \sum_{x \in f^{-1}(y)} \phi(x) \right),
 \end{aligned}$$

so we have proved (2.7).

Now we must show that  $\text{BAff}(f)$  is a linear positive unital map. The linearity follows from the pointwiseness of the operations defined on  $\text{BAff}(X)$ . To show  $\text{BAff}(f)$  is positive, let  $b \in \text{BAff}(Y)_+$ , *i.e.*  $b(y) \geq 0$  for all  $y \in Y$ . Then

$$\text{BAff}(f)(b)(x) = b(f(x)) \geq 0$$

for all  $x \in X$ , so  $\text{BAff}(f)(b) \in \text{BAff}(X)$ .

Since

$$\text{BAff}(f)(u)(x) = u(f(x)) = 1$$

we have  $\text{BAff}(f)(u) = u$ , so  $\text{BAff}(f)$  is unital.

Finally,  $\text{BAff}$  preserves identity maps because  $a \circ \text{id}_X = a$  and preserves composition by associativity of composition.  $\square$

If we have a bounded linear functional  $\phi : E \rightarrow \mathbb{R}$  where  $E$  is a pre-base-norm space, then  $\phi|_{B^{\mathcal{D}}(E)}$  is a bounded affine function on  $B^{\mathcal{D}}(E)$ . This defines

a restriction map  $\rho_E : E^* \rightarrow \text{BAff}(B^{\mathcal{D}}(E))$ . In addition to its Banach space structure as a dual space, we can define a positive cone

$$E_+^* = \{\phi \in E^* \mid \forall x \in E_+ \cdot \phi(x) \geq 0\},$$

which is a cone, and not merely a wedge, by Lemma 0.3.8, and we can define a unit element  $\tau$ , as by Lemma 2.2.4  $\tau \in E^*$  and as it is positive it defines an element of  $E_+^*$ .

We can then prove a slight generalization of [6, Proposition 1.11], that also follows from [91, Theorem 1 (ii)].

**Proposition 2.4.17.** *The map  $\rho_E : E^* \rightarrow \text{BAff}(B^{\mathcal{D}}(E))$  is a linear isomorphism preserving the positive cone and unit both ways. Therefore  $E^*$  is a Banach order-unit space for any pre-base-norm space  $E$ , and the closed unit ball as a dual space is exactly the interval  $[-\tau, \tau]$ . In the case that  $E$  has  $\sigma$ -convex base, such as when  $E \in \mathbf{BBNS}$ ,  $\rho_E$  defines an isomorphism  $E^* \rightarrow \text{BAff}(B^{\mathcal{D}\infty}(E))$ .*

*Proof.* Let  $a \in E^*$ . We first show that  $a|_{B^{\mathcal{D}}(E)} \in \text{BAff}(B^{\mathcal{D}}(E))$ . As  $B^{\mathcal{D}}(E) \subseteq \text{Ball}(E)$  (Lemma 0.1.6),  $a$  is bounded on  $B^{\mathcal{D}}(E)$ . If  $\phi \in \mathcal{D}(B^{\mathcal{D}}(E))$ , then

$$\begin{aligned} a(\alpha_{B^{\mathcal{D}}(E)}(\phi)) &= a(B^{\mathcal{D}}(\epsilon_E)(\phi)) = a\left(\sum_{x \in B^{\mathcal{D}}(E)} \phi(x) \cdot x\right) \\ &= \sum_{x \in B^{\mathcal{D}}(E)} \phi(x) \cdot a(x), \end{aligned}$$

so  $\rho_E(a) \in \text{BAff}(E)$ . The map  $\rho_E$  is linear because the addition is pointwise. It is injective because if  $\rho_E(a) = \rho_E(b)$  then  $a$  and  $b$  agree on  $B^{\mathcal{D}}(E)$ , and as  $E$  is the span of  $B^{\mathcal{D}}(E)$ ,  $a = b$ .

To show that  $\rho_E$  is surjective, let  $a \in \text{BAff}(B^{\mathcal{D}}(E))$ . Since it is bounded, there exists an  $\alpha \in \mathbb{R}_{\geq 0}$  such that  $|a(x)| \leq \alpha$  for all  $x \in B^{\mathcal{D}}(E)$ . Reusing the affine isomorphism  $i_\alpha$  from (2.6), we have  $i_\alpha \circ a : B^{\mathcal{D}}(E) \rightarrow B^{\mathcal{D}}(\mathbb{R}^2)$ , and so by Proposition 2.4.8 it extends to a trace-preserving map  $\overline{i_\alpha \circ a} : E \rightarrow \mathbb{R}^2$ . We can define a linear map

$$\begin{aligned} p_\alpha : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ p_\alpha(x, y) &= -\alpha x + \alpha y, \end{aligned}$$

and as this is a map between finite dimensional spaces, this is bounded. There-

fore  $p_\alpha \circ \overline{i_\alpha \circ a} \in E^*$ . Now, let  $x \in B^\mathcal{D}(E)$  in the following

$$\begin{aligned} \rho_E(p_\alpha \circ \overline{i_\alpha \circ a})(x) &= p_\alpha(i_\alpha(a(x))) = p_\alpha\left(\frac{\alpha - a(x)}{2\alpha}, \frac{a(x) + \alpha}{2\alpha}\right) \\ &= \frac{a(x) - \alpha}{2} + \frac{a(x) + \alpha}{2} = a(x). \end{aligned}$$

This proves that  $\rho_E$  is a linear bijection.

We chose  $\tau$  to be the unit element of  $E^*$ . Now, for any element  $x \in B^\mathcal{D}(E)$ ,  $\tau(x) = 1$ , so  $\rho_E(\tau) = u$ . Now, if  $a \in E_+^*$ , then since  $B^\mathcal{D}(E) \subseteq E_+$ , we have that  $\rho_E(a)(x) \geq 0$  for all  $x \in B^\mathcal{D}(E)$ , and so  $\rho_E(a) \in \text{BAff}(B^\mathcal{D}(E))_+$ . If, on the other hand,  $\rho_E(a) \in \text{BAff}(B^\mathcal{D}(E))$ , then, as each  $x \in E_+$  can be expressed as  $\alpha x'$  for  $x' \in B^\mathcal{D}(E)$  and  $\alpha \in \mathbb{R}_{\geq 0}$ , so

$$a(x) = a(\alpha x') = \alpha a(x') \geq 0$$

and  $a \in E_+^*$ . Therefore  $E^*$  is a Banach order-unit space and  $\rho_E$  an isomorphism in **BOUS** between  $E^*$  and  $\text{BAff}(B^\mathcal{D}(E))$ .

All that is left to prove is that the usual unit ball of  $E^*$  as a dual space coincides with  $[-\tau, \tau]$ . Suppose  $a \in \text{Ball}(E^*)$ . Then for all  $x \in \text{Ball}(E)$ , we have  $|a(x)| \leq 1$ , or  $1 \leq a(x) \leq 1$ . By Lemma 0.1.6,  $B^\mathcal{D}(E) \subseteq \text{Ball}(E)$ , so for all  $x \in B^\mathcal{D}(E)$  we have  $-u(x) = -1 \leq \rho_E(a)(x) \leq 1 = u(x)$ . As  $\rho_E$  is a poset isomorphism and  $\rho_E(\tau) = u$ , we have shown  $-\tau \leq a \leq \tau$ .

For the other direction, suppose that  $-\tau \leq a \leq \tau$ . Suppose for a contradiction that there exists some  $x \in \text{Ball}(E)$  such that  $|a(x)| > 1$ , taking  $\alpha = |a(x)|$ . So

$$\left\| \frac{2}{1+\alpha} x \right\| = \frac{2}{1+\alpha} \|x\| = \frac{2}{1+\alpha} < 1.$$

By Lemma 0.1.6,  $\frac{2}{1+\alpha} x \in \text{absco}(B^\mathcal{D}(E))$ . If  $B^\mathcal{D}(E) = \emptyset$ , then  $a(x) = 0$  for all  $x \in E$  so we have a contradiction. Therefore we reduce to the case that  $B^\mathcal{D}(E) \neq \emptyset$ , and therefore  $\frac{2}{1+\alpha} x = \beta x_+ + (1-\beta)x_-$ , with  $x_+, x_- \in B^\mathcal{D}(E)$  and  $\beta \in [0, 1]$ . By the assumption on  $a$ , we have  $-1 \leq a(x_\pm) \leq 1$ , or  $|a(x_\pm)| \leq 1$ . Therefore

$$\begin{aligned} \left| a\left(\frac{2}{1+\alpha} x\right) \right| &= |a(\beta x_+ + (1-\beta)x_-)| \leq \beta |a(x_+)| + (1-\beta) |a(x_-)| \\ &\leq \beta + 1 - \beta = 1. \end{aligned}$$

By linearity of  $a$ , this implies

$$|a(x)| \leq \frac{1+\alpha}{2},$$

but this contradicts  $|a(x)| = \alpha > 1$ .

Finally, the statement for  $\text{BAff}(B^{\mathcal{D}\infty}(E))$  follows by Lemma 2.4.14.  $\square$

Given a trace-preserving map  $f : E \rightarrow F$ , we can define  $f^* : F^* \rightarrow E^*$  as  $f^*(b) = b \circ f$ .

**Theorem 2.4.18.** *This definition makes  $-^*$  a functor  $\mathbf{PreBNS} \rightarrow \mathbf{BOUS}^{\text{op}}$ , and  $\rho$  is a natural isomorphism  $-^* \Rightarrow \text{BAff} \circ B^{\mathcal{D}}$ , and also  $-^* \Rightarrow \text{BAff} \circ B^{\mathcal{D}\infty}$ .*

*Proof.* We do the proof only for  $\text{BAff} \circ B^{\mathcal{D}}$ , using Lemma 2.4.14 for  $B^{\mathcal{D}\infty}$ .

If we show that the naturality diagram commutes, and then that this implies that the definition of  $-^*$  on maps is a functor and  $\rho$  is a natural transformation. The diagram in question, for  $f \in \mathbf{PreBNS}(E, F)$ , is

$$\begin{array}{ccc} F^* & \xrightarrow{\rho_F} & \text{BAff}(B^{\mathcal{D}}(F)) \\ f^* \downarrow & & \downarrow \text{BAff}(B^{\mathcal{D}}(f)) \\ E^* & \xrightarrow{\rho_E} & \text{BAff}(B^{\mathcal{D}}(E)). \end{array}$$

To show that this commutes, let  $b \in F^*$  and  $x \in B^{\mathcal{D}}(E)$ . Then

$$\begin{aligned} \rho_E(f^*(b))(x) &= f^*(b)(x) = b(f(x)) = \rho_F(b)(B^{\mathcal{D}}(f)(x)) \\ &= \text{BAff}(B^{\mathcal{D}}(f))(\rho_F(b))(x). \end{aligned}$$

The commutativity of the diagram implies that if  $f \in \mathbf{PreBNS}(E, F)$ ,  $\rho_E^{-1} \circ \text{BAff}(B^{\mathcal{D}}(f)) \circ \rho_F = f^*$ . Since each of the maps composing to give  $f^*$  is linear, positive and unital, this proves that  $f^*$  is. We can therefore show

$$\begin{aligned} \text{id}_E^* &= \rho_E^{-1} \circ \text{BAff}(B^{\mathcal{D}}(\text{id}_E)) \circ \rho_E \\ &= \rho_E^{-1} \circ \rho_E && \text{BAff}, B^{\mathcal{D}} \text{ functors} \\ &= \text{id}_{E^*}. \end{aligned}$$

In the case that  $f \in \mathbf{PreBNS}(E, F)$  and  $g \in \mathbf{PreBNS}(F, G)$ , we have

$$\begin{aligned} (g \circ f)^* &= \rho_E^{-1} \circ \text{BAff}(B^{\mathcal{D}}(g \circ f)) \circ \rho_G \\ &= \rho_E^{-1} \circ \text{BAff}(B^{\mathcal{D}}(f)) \circ \text{BAff}(B^{\mathcal{D}}(g)) \circ \rho_G \\ &= \rho_E^{-1} \circ \text{BAff}(B^{\mathcal{D}}(f)) \circ \rho_F \circ \rho_F^{-1} \circ \text{BAff}(B^{\mathcal{D}}(g)) \circ \rho_G \\ &= f^* \circ g^*, \end{aligned}$$

showing that  $-^*$  is a contravariant functor. The diagram we started with then shows that  $\rho_E$  is natural.  $\square$

## 2.5 Dualities with Order-Unit Spaces

In this section, we will prove certain categorical duality results for base-norm and order-unit spaces. We start with a dual adjunction between **PreBNS** and **OUS**, related to that between  $\mathcal{EM}(\mathcal{D})$  and **EA** in [52, Theorem 17]. We then see how this adjunction restricts to an equivalence. As the equivalence derived from it is not entirely satisfactory, we will see in the next chapter how to adapt it to define dual categories for **BBNS** and **BOUS**.

### 2.5.1 The Dual Adjunction

In this section, we define a functor  $F : \mathbf{PreBNS} \rightarrow \mathbf{BOUS}^{\text{op}}$ , and another  $G : \mathbf{OUS}^{\text{op}} \rightarrow \mathbf{BBNS}$  such that, when composed with the inclusions  $\mathbf{BOUS} \hookrightarrow \mathbf{OUS}$  and  $\mathbf{BBNS} \hookrightarrow \mathbf{PreBNS}$ ,  $F$  is a left adjoint to  $G$ . We use the same definition to get functors  $F : \mathbf{PreBNS}_{\leq 1} \rightarrow \mathbf{BOUS}_{\leq 1}^{\text{op}}$  and  $G : \mathbf{OUS}_{\leq 1}^{\text{op}} \rightarrow \mathbf{BBNS}_{\leq 1}$ ,  $F$  left adjoint to  $G$ . The simplest way to prove this adjunction is to use the unit-counit definition of an adjunction (Theorem 0.4.1 (v)).

In the case of trace-preserving maps, we have already seen  $F$ . It is  $-^* : \mathbf{PreBNS} \rightarrow \mathbf{BOUS}^{\text{op}}$  from the previous section. We give below the definition for trace-reducing maps, which looks identical to the trace-preserving definition:

$$\begin{aligned} F(X, X_+, \tau) &= (X^*, X_+^*, \tau) \\ F(f) : F(Y, Y_+, \sigma) &\rightarrow F(X, X_+, \tau) \\ F(f)(a) &= a \circ f, \end{aligned}$$

where  $f : (X, X_+, \tau) \rightarrow (Y, Y_+, \sigma)$  and  $a \in Y^*$ .

To define  $G$ , we need a standard theorem.

**Theorem 2.5.1** (Ellis [34]). *If  $(A, A_+, u)$  is an (archimedean) order-unit space,  $(A^*, A_+^*, \text{ev}(u))$  is a (radially compact) Banach base-norm space. The base-norm on  $A^*$  agrees with the usual dual norm.*

*Proof.* In the case that  $A \neq 0$ , we refer the reader to [6, Theorem 1.19] [4, Theorem II.1.15], each of which proves that the dual is a base-norm space in the Alfsen-Shultz sense, and therefore a base-norm space by Proposition 2.2.20. If we use [6, Theorem 1.19] as a reference, we must additionally use the “norm duality” part of “order and norm duality” in [6, paragraph after Lemma 1.22] and [6, Corollary 1.27]. The fact that the dual is a Banach space follows from the fact that the dual of a normed space is complete, using the fact that the norm coincides with the usual dual norm. In the case that  $A = (\{0\}, \{0\}, 0)$  its dual space is  $(\{0\}, \{0\}, 0)$ , a Banach base-norm space, with empty base.  $\square$

We can now define  $G$  on objects and maps quite similarly to  $F$ :

$$\begin{aligned} G(A, A_+, u) &= (A^*, A_+^*, \text{ev}(u)) \\ G(f) : G(B, B_+, v) &\rightarrow G(A, A_+, u) \\ G(f)(\phi) &= \phi \circ f, \end{aligned}$$

where  $f : (A, A_+, u) \rightarrow (B, B_+, v)$  is a map in **OUS**, corresponding to a map in the opposite direction in **OUS**<sup>op</sup>, and  $\phi \in B^*$ .

**Proposition 2.5.2.**  *$F$  and  $G$  are functors.*

*Proof.* We have already shown in the trace-preserving case that  $F$  is a functor in Theorem 2.4.18. We also have the general result that  $F(X, X_+, \tau)$  is a Banach order-unit space (Proposition 2.4.17), and by Theorem 2.5.1,  $G(A, A_+, u)$  is a Banach base-norm space. We now check that  $F$  and  $G$  have the correct type on morphisms, for  $F$  only in the trace-reducing case.

Let  $f : (X, X_+, \tau) \rightarrow (Y, Y_+, \sigma)$  be a trace-reducing map, and  $a \in Y^*$ . We need to show that  $F(f)(a) = a \circ f \in X^*$  and that  $F(f)$  is positive and unital. Since  $f$  is bounded by Proposition 2.2.12,  $a \circ f$  is a bounded linear functional  $X \rightarrow \mathbb{R}$ , hence an element of  $X^*$ . If we let  $g : (A, A_+, u) \rightarrow (B, B_+, v)$  be a unital or subunital map,  $g$  is bounded by Proposition 1.2.8, it is then the case that  $\phi \circ g$  is bounded if  $\phi$  is, and hence is an element of  $G(A, A_+, v)$ .

The proofs that  $F(f)$  and  $G(g)$  are positive are nearly identical to each other, so we will only give the proof for  $F(f)$  explicitly. We must show that if  $a \in Y_+^*$ ,  $F(f)(a) \in X_+^*$ . By the definition of the dual cone, this is equivalent to showing that  $\forall y \in Y_+. a(y) \geq 0$  implies  $\forall x \in X_+. F(f)(a)(x) \geq 0$ . We can show this as follows. Let  $x \in X_+$ . Then  $F(f)(a)(x) = a(f(x))$ . We have that  $f(x) \in Y_+$  by positivity of  $f$ , and therefore  $a(f(x)) \geq 0$  by positivity of  $a$ .

To show that  $F(f)$  is subunital when  $f$  is trace-reducing, we want to show that  $F(f)(\sigma) \leq \tau$  in  $X^*$ , i.e.  $\tau - F(f)(\sigma) \in X_+^*$ . So let  $x \in X_+$ , and:

$$(\tau - F(f)(\sigma))(x) = \tau(x) - \sigma(f(x)) \geq 0,$$

by the definition of trace-reducing for  $f$ .

To show that  $G(g)$  is trace-preserving when  $g$  is unital, we must show that  $\text{ev}(u) \circ G(g) = \text{ev}(v)$ . We do so as follows. Let  $\phi \in B^*$ . Then

$$\begin{aligned} (\text{ev}(u) \circ G(g))(\phi) &= \text{ev}(u)(G(g)(\phi)) \\ &= G(g)(\phi)(u) \\ &= \phi(g(u)) && \text{so since } g \text{ is unital} \\ &= \phi(v) = \text{ev}(v)(\phi). \end{aligned}$$



To show that  $G(g)$  is trace-reducing when  $g$  is subunital, we want to show that  $\text{ev}(u) \circ G(g) \leq \text{ev}(v)$ , i.e.  $\text{ev}(v) - \text{ev}(u) \circ G(g) \in B_+^{**}$ . Let  $\phi \in B_+^*$ , and we have that

$$\begin{aligned} (\text{ev}(v) - \text{ev}(u) \circ G(g))(\phi) &= \text{ev}(v)(\phi) - \text{ev}(u)(G(g)(\phi)) \\ &= \phi(v) - \text{ev}(u)(\phi \circ g) \\ &= \phi(v) - \phi(g(u)) \\ &= \phi(v - g(u)). \end{aligned}$$

Since  $g$  is subunital,  $v - g(u) \in B_+$ , and since  $\phi \in B_+^*$ , we have  $\phi(v - g(u)) \geq 0$ . This implies  $G(g)$  is trace-reducing.

This establishes that  $F$  and  $G$  are defined correctly. The fact that they preserve identity arrows follows from the identity law for composition for linear maps, and the fact that they preserve composition follows from the associativity of composition for linear maps.  $\square$

Now we can move on to the definition of the unit and counit.

Taking  $(X, X_+, \sigma)$  to be a pre-base-norm space, we define

$$\begin{aligned} \eta_X : X &\rightarrow GF(X) \\ \eta_X(x)(a) &= a(x), \end{aligned}$$

where  $a \in F(X)$ .

For  $(A, A_+, u)$  an order-unit space, we want to define  $\epsilon_A : FG(A) \rightarrow A$  in **OUS**<sup>op</sup>, which means that in **OUS** we define

$$\begin{aligned} \epsilon_A : A &\rightarrow FG(A) \\ \epsilon_A(a)(\phi) &= \phi(a), \end{aligned}$$

where  $\phi \in G(A)$ .

**Proposition 2.5.3.** *The maps  $\eta_X$  and  $\epsilon_A$  are well-defined and are natural transformations.*

*Proof.* We must first show that  $\eta_X$  and  $\epsilon_A$  are well-defined and are positive. Since the proofs are very similar, we will only state it explicitly for  $\eta_X$ .

We must show that  $\eta_X(x) \in GF(X) = X^{**}$ , which is to say that if  $\phi \in X^*$  and  $\|\phi\| \leq 1$ , we have  $\|\eta_X(x)(\phi)\| \leq \alpha$  for some  $\alpha \in [0, \infty)$  (the norms on  $X^*$  and  $X^{**}$  agree with the usual definition of the dual norms by Proposition 2.2.12, or for  $\epsilon_A$ , use Theorem 2.5.1 instead). If  $\|x\| = 0$ , then  $x = 0$ , and

$\eta(x)(\phi) = \phi(0) = 0$  so we can take  $\alpha = 0$ . So we now assume that  $\|x\| > 0$ . Then

$$\eta(x)(\phi) = \phi(x) = \phi\left(\|x\| \frac{x}{\|x\|}\right) = \|x\| \phi\left(\frac{x}{\|x\|}\right).$$

Because  $\|\phi\| \leq 1$ , we have that  $\phi\left(\frac{x}{\|x\|}\right) \leq 1$ , giving us

$$\eta(x)(\phi) \leq \|x\|.$$

We now show that  $\eta_X$  is positive, *i.e.* that if  $x \in X_+$ ,  $\eta_X(x) \in X_+^{**}$ . If  $a \in X_+^*$ , then  $\eta_X(x)(a) = a(x)$ , which is positive because  $x \in X_+$  and  $a \in X_+^*$ . Since this works for an arbitrary element of  $X_+^*$ , we have that  $\eta_X(x) \in X_+^{**}$ .

We now show separately that  $\eta_X$  is trace-preserving (and hence also trace-reducing) and that  $\epsilon_A$  is unital (and hence also subunital). We start with  $\eta_X$ . Since  $GF(X)$ 's trace is  $\text{ev}(\tau)$ , what we want to show is that  $\text{ev}(\tau) \circ \eta_X = \tau$ . Taking  $x \in X$ , we have

$$\text{ev}(\tau)(\eta_X(x)) = \eta_X(x)(\tau) = \tau(x),$$

as required.

For  $\epsilon_A$ , we have that the unit of  $FG(A)$  is  $\text{ev}(u)$ , so we want to show  $\epsilon_A(u) = \text{ev}(u)$ . If we take  $\phi \in A^*$ , we have

$$\epsilon_A(u)(\phi) = \phi(u) = \text{ev}(u)(\phi).$$

Finally, we must show that  $\eta_X$  and  $\epsilon_A$  define natural transformations. The proof is again very similar in each case so we shall only give the proof for  $\eta$ . Let  $f : (X, X_+, \tau) \rightarrow (Y, Y_+, \sigma)$  be trace-preserving (or trace-reducing). We want to show that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ GF(X) & \xrightarrow{GF(f)} & GF(Y) \end{array}$$

commutes. That is to say, if  $x \in X$ , and  $b \in F(Y) = Y^*$ , we want to show that

$$\eta_Y(f(x))(b) = GF(f)(\eta_X(x))(b).$$

We proceed as follows:

$$\begin{aligned} G(F(f))(\eta_X(x))(b) &= (\eta_X(x) \circ F(f))(b) = \eta_X(x)(F(f)(b)) = \eta_X(x)(b \circ f) \\ &= b(f(x)) = \eta_Y(f(x))(b). \end{aligned}$$

We did not use the trace-preservation, so this naturality argument works equally well for trace-reducing maps. Similarly, it also holds for subunital maps.  $\square$

We can now finally prove  $F \dashv G$ . This is original, in this categorical form, as is the equivalence derived from it.

**Theorem 2.5.4.** *The functor  $F : \mathbf{PreBNS} \rightarrow \mathbf{OUS}^{\text{op}}$  is a left adjoint to the functor  $G : \mathbf{OUS}^{\text{op}} \rightarrow \mathbf{PreBNS}$ , in both the case of trace-preserving/unital and trace-reducing/subunital maps.*

*Proof.* We want to show that the following diagrams commute, where  $A \in \mathbf{OUS}$  and  $X \in \mathbf{PreBNS}$ :

$$\begin{array}{ccc} GA & \xrightarrow{\eta_{GA}} & GF GA \\ & \searrow \text{id}_{GA} & \downarrow G\epsilon_A \\ & & GA \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FGFX \\ & \searrow \text{id}_{FX} & \downarrow \epsilon_{FX} \\ & & FX \end{array}$$

The right-hand diagram is written in  $\mathbf{OUS}^{\text{op}}$ . If it is written in  $\mathbf{OUS}$ , with the arrows turned back to normal, the proof that it commutes is virtually the same as the proof that the left-hand diagram commutes. Therefore we will only give the proof explicitly that the left-hand diagram commutes. The proof for  $\mathbf{OUS}_{\leq 1}$  and  $\mathbf{PreBNS}_{\leq 1}$  is identical to the proof for  $\mathbf{OUS}$  and  $\mathbf{PreBNS}$  as the definitions of the functors and natural transformations are identical.

With that out of the way, we have to show that for  $\phi \in G(A)$ , we have

$$G(\epsilon_A)(\eta_{GA}(\phi)) = \phi.$$

We can do this by evaluating the left-hand side at an arbitrary element  $a \in A$ :

$$\begin{aligned} G(\epsilon_A)(\eta_{GA}(\phi))(a) &= (\eta_{GA}(\phi) \circ \epsilon_A)(a) = \eta_{GA}(\phi)(\epsilon_A(a)) = \epsilon_A(a)(\phi) \\ &= \phi(a). \end{aligned}$$

$\square$

As with any adjunction, we can consider those objects such that the unit and counit are isomorphisms. We can define *reflexive order-unit spaces* to be order-unit spaces  $A$  such that  $\epsilon_A$  is an isomorphism of order-unit spaces and *reflexive base-norm spaces* to be pre-base-norm spaces  $X$  such that  $\eta_X$  is an isomorphism of pre-base-norm spaces. We can define  $\mathbf{RBNS}$  to be the full subcategory of  $\mathbf{PreBNS}$  on reflexive base-norm spaces and  $\mathbf{ROUS}$  to be the

full subcategory of **OUS** on reflexive order-unit spaces. Recall that a normed space  $E$  is called *reflexive* if the evaluation map  $E \rightarrow E^{**}$  is a bijection [20, III Definition 11.2].

**Proposition 2.5.5.** *A pre-base-norm space is reflexive if and only if its underlying normed space is reflexive and it is a base-norm space. An order-unit space is reflexive iff its underlying normed space is reflexive. The functors  $F$  and  $G$ , when restricted to **RBNS** and **ROUS** respectively, form an adjoint equivalence  $\mathbf{RBNS} \simeq \mathbf{ROUS}^{\text{op}}$ .*

*Proof.* On the underlying normed spaces, the maps  $\epsilon_A$  and  $\eta_E$  are the evaluation maps. Therefore reflexivity of the underlying normed spaces is necessary in both cases. As dual cones are always weakly closed (Lemmas 0.3.5 and 0.3.7) they are closed in the finer norm topology as well, it is also necessary that a reflexive pre-base-norm space be a base-norm space, otherwise the inverse of  $\eta_E$  would not be positive. We have therefore shown the necessity in both cases.

For sufficiency, observe first that any bijective unital map of order-unit spaces has unital inverse, and every bijective trace-preserving map of pre-base-norm spaces has trace-preserving inverse. Both a base-norm space (by definition) and an order-unit space (by Lemma A.5.3) have a closed positive cone. We therefore only need to show that for a reflexive Banach space  $E$  with a closed cone  $E_+$ , the inverse of the evaluation mapping  $E^{**} \rightarrow E$  is positive, where  $E^{**}$  is given the double dual cone. This is equivalent to showing that  $\text{ev}^{-1}(E_+^{**}) \subseteq E_+$ , as we already know the opposite inclusion holds by Proposition 2.5.3. So let  $x \in E$  be such that  $\text{ev}(x) \in E_+^{**}$ . Then by expanding the definition of the dual cone, we have that for all  $\phi \in E_+^*$ ,  $\text{ev}(x)(\phi) \geq 0$ . As  $\text{ev}(x)(\phi) = \phi(x)$ , this is equivalent to  $\phi(x) \geq 0$ . Then Lemma 0.3.15 gives us  $x \in E_+$  because it is closed.

Therefore  $F$  and  $G$  form an adjoint equivalence  $\mathbf{ROUS}^{\text{op}} \simeq \mathbf{RBNS}$  by the “unity of opposites” – the triangle laws for an adjunction imply that if  $\eta_E$  is an isomorphism,  $\epsilon_{F(E)}$  is too, and similarly  $\epsilon_A$  an isomorphism implies  $\eta_{G(A)}$  is an isomorphism [76, Part 0, Proposition 4.2].  $\square$

A normed space is reflexive iff its unit ball is compact in the weak topology [20, V Theorem 4.2]. Therefore every finite-dimensional normed space is reflexive because the unit ball is compact by the Heine-Borel theorem. So the finite dimensional base-norm spaces (excluding pre-base-norm spaces) and finite dimensional order-unit spaces are all reflexive. There are also reflexive infinite-dimensional order-unit spaces such as the *spin factors* [7, Proposition

3.38]. The corresponding base-norm spaces are those arising from taking the unit ball of a Hilbert space as an element of **CBConv** [7, Proposition 5.51].

However, every reflexive  $C^*$ -algebra is finite dimensional [118, I.11 Exercise 2]. In particular, this means that  $C(X)$  is reflexive iff  $X$  is finite, so we cannot include examples such as  $C([0, 1])$  in this duality. Similarly,  $B(\mathcal{H})$  is an infinite-dimensional  $C^*$ -algebra if  $\mathcal{H}$  is infinite-dimensional<sup>6</sup>. As the duality  $\mathbf{RBNS} \simeq \mathbf{ROUS}^{\text{op}}$  is not general enough to include all the useful examples, we adapt it into a pair of dualities in chapter 3.

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<sup>6</sup>As it must be to represent the canonical commutation relations [82] or have nontrivial unitary representations of the Lorentz group [71, §XVI.1].



# Chapter 3

## Smith Spaces

### 3.1 Introduction

The purpose of this chapter is to show how the dual adjunction between base-norm spaces and order-unit spaces in the previous chapter can be converted into two dualities. In the previous chapter we saw that we could produce a duality through the use of reflexive spaces, so in this chapter we first look at a way to make every Banach space “reflexive”, by considering a different dual topology. This leads us to consider Smith spaces. A space  $E$  is a *Smith space* if it is linearly homeomorphic to the dual space of some Banach space  $F$ , given the *bounded weak-\** topology or equivalently the topology of uniform convergence on precompact sets. The term was introduced by Akbarov [2, Example 4.6], naming them after M. F. Smith who published a paper on Pontryagin duality for Banach spaces [113, Theorem 2]. Akbarov gives in [2, Proposition 4.7] and [3, Proposition 1.2] an intrinsic characterization of Smith spaces as locally convex spaces  $E$  satisfying three criteria:

- (i)  $E$  is complete.
- (ii)  $E$  is compactly generated.
- (iii) There is a compact absolutely convex set in  $E$  that absorbs every compact subset (a *universal compact set*).

In this chapter we show that the first condition can be dropped and the other two relaxed so a space can be confirmed to be Smith more easily (Proposition

3.2.9). We then prove a categorical equivalence (Theorem 3.2.22)

$$\mathbf{Ban}^{\text{op}} \simeq \mathbf{Smith},$$

though this is known, with Akbarov's definitions, from [2, Theorems 4.1, 4.2 and 4.11 (a)]. In fact, we would be able to do this using the more easily defined weak-\* topology instead of the bounded weak-\* topology, but we know of no characterization of such spaces, except by first characterizing the bounded weak-\* topology. Another dual category to Banach spaces is the category of Waelbroeck spaces, as used in [19, Chapter I, Theorem 2.8]. These spaces only put a topology on the unit ball and so avoid having to make a distinction between the weak-\* and bounded weak-\* topologies, although the difficulties are shifted elsewhere as one does not, *prima facie*, have a linear topology.

We then turn the duality from Theorem 2.5.4 into two dualities, by choosing either one side to be Smith spaces and the other to be Banach spaces, in each case. That is to say, we define Smith base-norm and order-unit spaces, and prove

$$\mathbf{SBNS} \simeq \mathbf{BOUS}^{\text{op}}$$

$$\mathbf{BBNS} \simeq \mathbf{SOUS}^{\text{op}}.$$

The first duality includes all Banach order-unit spaces, and the second all Banach base-norm spaces, so reflexivity is no longer required.

We can extend these equivalences to two squares of equivalences:

$$\begin{array}{ccc} \mathbf{BOUS}^{\text{op}} & \xrightleftharpoons[G^{\sigma}]{F^{\beta}} & \mathbf{SBNS} \\ \tau \updownarrow [0,1]- & & \text{Emb} \updownarrow B \\ \mathbf{BEMod}^{\text{op}} & \xrightleftharpoons[\text{CAff}(-,[0,1])]{\text{Stat}} & \mathbf{CCL} \end{array} \quad (3.1)$$

$$\begin{array}{ccc} \mathbf{SOUS}^{\text{op}} & \xrightleftharpoons[G^{\beta}]{F^{\sigma}} & \mathbf{BBNS} \\ \tau \updownarrow [0,1]- & & \text{Emb} \updownarrow B \\ \mathbf{CEMod}^{\text{op}} & \xrightleftharpoons[\text{BAff}(-,[0,1])]{\text{CStat}} & \mathbf{CBConv} \end{array} \quad (3.2)$$

Where  $\mathbf{CEMod}$  and  $\mathbf{BEMod}$  are categories of effect modules to be used as predicates, and  $\mathbf{CBConv}$  and  $\mathbf{CCL}$  are categories of convex sets to be used as state spaces. The square (3.1) can be viewed as a summary of Kadison



duality (see [55] and [64]), the equivalence between **BOUS** and **BEMod** (see [59, Proposition 11] and [48]), and the fact that a base-norm space is a dual space iff it can be given a locally convex topology in which its base is compact [34, Theorem 3]. The square (3.2), on the other hand, is mostly original, the only preceding result being [34, Theorem 6], that an order-unit space is the dual of a base-norm space iff its unit interval is compact in some locally convex topology. We can use these squares and the relationship between  $C^*$ -algebras and order-unit spaces to produce a state-and-effect triangle each for  $C^*\mathbf{Alg}_{PU}$  and  $W^*\mathbf{Alg}_{PU}$ .

In the next chapter we will see how to express the category **CCL** in terms of Eilenberg-Moore algebras, and so inherit the convenient properties thereof. In this chapter, we do show that the category **CBConv** is a reflective subcategory of  $\mathcal{EM}(\mathcal{D}_\infty)$  and  $\mathcal{EM}(\mathcal{D})$ . This shows how we can take the bare minimum for a structure of probabilistic mixtures on a set and “freely” construct a Banach base-norm space from it.

In Section 3.5 we show what happens if we combine the dualities in this chapter with the adjunction in Subsection 2.5.1. The resulting adjunctions are related to enveloping  $W^*$ -algebras and also to Semadeni’s universal compactification for convex sets.

## 3.2 Smith Spaces

*The purpose of this section is essentially to prove Theorem 3.2.22 and Corollary 3.2.23 with the laxer definition of Smith space used here. This is done by combining well-known results of functional analysis, so we make no claim to originality for the intermediate results used, but make them explicit for the benefit of the reader.*

Let  $E$  be a normed space and  $E^*$  its dual space. We have already seen that  $E^*$  can be given the dual norm, which defines one topology, and it can also be given the weak- $*$  (or  $\sigma(E^*, E)$ ) topology. We now concern ourselves with a third topology, in between these, the *bounded weak- $*$*  topology (see [28, Definition V.5.3, Corollary V.5.5]). This topology is defined to be the finest topology agreeing with  $\sigma(E^*, E)$  on bounded sets, *i.e.* a set  $O \subseteq E^*$  is open if for all  $\alpha \in (0, \infty)$  there exists a  $\sigma(E^*, E)$ -open  $O'$  such that

$$O \cap \alpha U = O' \cap \alpha U,$$

where  $U$  is the unit ball of  $E^*$  with respect to its usual norm.

On the dual of a Banach space  $E$ , the bounded weak- $*$  topology is also the same as the polar topology for compact subsets of  $E$  ([8, Chapter 1,

Theorem 2.2]). Smith spaces are related to the circle of ideas around the Krein-Šmulian theorem, the Banach-Dieudonné theorem and Grothendieck's completeness theorem.

A *barrel* in a locally convex space  $(E, \mathcal{T})$  is a subset  $B \subseteq E$  that is absolutely convex, closed, and absorbent. To avoid confusion, we state here that a *barrelled* space is one in which every barrel is a zero neighbourhood, but that every locally convex space contains several barrels, whether or not it is barrelled.

We now give our definition of a Smith space. A *Smith space*  $(E, \mathcal{T}, B)$  is a locally convex space  $E$ , the topology being  $\mathcal{T}$ , and a compact barrel  $B$ , such that  $\mathcal{T}$  is the finest topology agreeing with  $\mathcal{T}$  on all the subsets  $\alpha B$  for  $\alpha \in \mathbb{R}_{>0}$ . The category **Smith** of Smith spaces has continuous linear maps as morphisms, and the category **Smith**<sub>1</sub> is the subcategory of maps between Smith spaces  $f : (E, \mathcal{T}, B) \rightarrow (F, \mathcal{S}, C)$  such that  $f(B) \subseteq C$ .

At this stage, we can show that closed subspaces of Smith spaces are Smith.

**Lemma 3.2.1.** *If  $(E, \mathcal{T}, B)$  is a Smith space,  $F \subseteq E$  a closed linear subspace, then  $(F, \mathcal{T}|_F, B \cap F)$  is a Smith space.*

*Proof.* We first show that  $B \cap F$  is a compact barrel. We use the name  $C = B \cap F$ . We have that  $\mathcal{T}|_F$  agrees with  $\mathcal{T}$  on  $F$ , so  $C$  is  $\mathcal{T}|_F$ -compact because  $C$  is a closed subspace of a compact space  $B$ . As  $C$  is the intersection of two absolutely convex sets, it is absolutely convex. To show that it is absorbent, let  $x \in F$ . As  $x \in E$ , there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $x \in \alpha B$ . Therefore

$$x \in (\alpha B) \cap F = \alpha B \cap \alpha F = \alpha(B \cap F) = \alpha C.$$

To show  $(F, \mathcal{T}|_F, C)$  is Smith, we only need to show that any set  $U \subseteq F$  such that for all  $\alpha \in \mathbb{R}_{>0}$  there exists  $U_\alpha \in \mathcal{T}|_F$   $U \cap \alpha C = U_\alpha \cap \alpha C$ , then  $U \in \mathcal{T}|_F$ . So let  $U$  be such a set. As  $U_\alpha \in \mathcal{T}|_F$ , there exists  $V_\alpha \in \mathcal{T}$  such that  $U_\alpha = V_\alpha \cap F$ . We first show that

$$(V_\alpha \cup (E \setminus F)) \cap \alpha B = (U \cup (E \setminus F)) \cap \alpha B \quad (3.3)$$

for all  $\alpha \in \mathbb{R}_{>0}$ .

We have

$$\begin{aligned}
 (V_\alpha \cup (E \setminus F)) \cap \alpha B &= ((V_\alpha \cap F) \cup (E \setminus F)) \cap \alpha B \\
 &= (V_\alpha \cap F \cap \alpha B) \cup ((E \setminus F) \cap \alpha B) \\
 &= (U_\alpha \cap \alpha C) \cup ((E \setminus F) \cap \alpha B) \\
 &= (U \cap \alpha C) \cup ((E \setminus F) \cap \alpha B) \\
 &= (U \cap F \cap \alpha B) \cup ((E \setminus F) \cap \alpha B) \\
 &= (U \cup (E \setminus F)) \cap \alpha B,
 \end{aligned}$$

proving (3.3).

As  $F$  is closed,  $E \setminus F$  is  $\mathcal{T}$ -open, so  $V_\alpha \cup (E \setminus F)$  is a  $\mathcal{T}$ -open set. As  $E$  is a Smith space, (3.3) shows that  $U \cup (E \setminus F)$  is  $\mathcal{T}$ -open. Therefore

$$U = (U \cup (E \setminus F)) \cap F$$

is a  $\mathcal{T}|_F$ -open set, so  $(F, \mathcal{T}|_F, C)$  is a Smith space.  $\square$

From what we have proven so far, it is not yet clear that there are any useful Smith spaces, or that Smithness can be verified usefully in practice. The rest of this section is dedicated to resolving these matters.

**Lemma 3.2.2.** *Let  $(E, \mathcal{T}, B)$  be a locally convex space  $(E, \mathcal{T})$  and a compact barrel  $B \subseteq E$ . The set  $B^\circ \subseteq E^*$  is radially compact and absorbent, and therefore defines a norm  $\|\cdot\|_{B^\circ}$  on  $E^*$ , of which  $B^\circ$  is the closed unit ball.*

*Proof.* We first show that  $B^\circ$  is radially bounded. Suppose for a contradiction that  $B^\circ$  is radially unbounded. Then it contains an element  $\phi \neq 0$  such that  $n\phi \in B^\circ$  for all  $n \in \mathbb{N}$ . Since  $B^\circ = B^{|\phi|}$  (Lemma 0.3.6), we see that for all  $n \in \mathbb{N}$  and  $x \in B$

$$|\langle n\phi, x \rangle| \leq 1 \Leftrightarrow |n\phi(x)| \leq 1 \Leftrightarrow |\phi(x)| \leq \frac{1}{n}$$

Therefore  $\phi(x) = 0$  for all  $x \in B$ . Since  $B$  is absorbent, its span is all of  $E$ , so  $\phi = 0$ , a contradiction.

Since  $B^\circ$  is a polar, it is  $\sigma(E^*, E)$ -closed, therefore the intersection of any line with  $B^\circ$  is closed, so it is radially compact.

To show that  $B^\circ$  is absorbent, let  $\phi \in E^*$ . As  $\phi$  is continuous, there is an absolutely convex 0-neighbourhood  $U \subseteq E$  such that  $\phi(U) \subseteq (-1, 1)$ . Since  $B$  is compact, it is bounded (Lemma 0.1.14), so there is an  $\alpha \in \mathbb{R}_{>0}$  such that  $B \subseteq \alpha U$ . Therefore  $\alpha^{-1}B \subseteq U$  and so  $\phi(\alpha^{-1}B) \subseteq (-1, 1)$ . This implies  $\phi \in (\alpha^{-1}B)^\circ = \alpha B^\circ$  (Lemma 0.3.11 (ii)).

By Lemma 0.1.5  $\|-\|_{B^\circ}$  is a norm, and by Lemma 0.1.7,  $B^\circ$  is the closed unit ball of  $\|-\|_{B^\circ}$ .  $\square$

In fact, as  $B$  is itself radially compact,  $\|-\|_B$  is a norm.

**Lemma 3.2.3.** *Let  $(E, \mathcal{T}, B)$  be a locally convex space with compact barrel  $B$ . The topology defined by the norm  $\|-\|_B$  is finer than  $\mathcal{T}$ .*

*Proof.* As the  $\|-\|_B$  topology and  $\mathcal{T}$  are both locally convex topologies, it suffices to show that every 0-neighbourhood for  $\mathcal{T}$  is a 0-neighbourhood for  $\|-\|_B$ . So let  $N$  be a 0-neighbourhood for  $\mathcal{T}$ , and  $U \subseteq N$  an open subset of  $N$  containing 0. As all compact sets are bounded (Lemma 0.1.14), we have that there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $B \subseteq \alpha U$ . Therefore  $\alpha^{-1}B \subseteq U \subseteq N$ , so  $N$  is a 0-neighbourhood for  $\|-\|_B$ .  $\square$

**Lemma 3.2.4.** *If  $(E, \mathcal{T}, B)$  is a locally convex space with  $B$  a compact barrel, the usual pairing  $\langle -, - \rangle$  between  $E$  and  $E^*$  is separately continuous for  $\mathcal{T}$  and  $\|-\|_{B^\circ}$ .*

*Proof.*

- For all  $\phi \in E^*$ ,  $\langle -, \phi \rangle : (E, \mathcal{T}) \rightarrow \mathbb{R}$  is continuous:  
Since  $\langle -, \phi \rangle = \phi$ , this follows from the definition of  $E^*$  as the continuous dual.
- For all  $x \in E$ ,  $\langle x, - \rangle : (E^*, \|-\|_{B^\circ}) \rightarrow \mathbb{R}$  is continuous:  
As  $E^*$  is normed, we only need to show that  $\langle x, - \rangle$  is bounded for each  $x \in E$ , i.e. that there exists some  $\alpha \in \mathbb{R}_{>0}$  such that  $\langle x, B^\circ \rangle \subseteq [-\alpha, \alpha]$ . Since  $B$  is absorbent, there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $\alpha^{-1}x \in B$ . We know by Lemma 0.3.6 that  $|\phi(\alpha^{-1}x)| \leq 1$  for all  $\phi \in B^\circ$ . Therefore  $|\langle x, \phi \rangle| = |\phi(x)| \leq \alpha$  for all  $\phi \in B^\circ$ , which gives us that  $\langle x, B^\circ \rangle \subseteq [-\alpha, \alpha]$ , as required.  $\square$

We need the following small lemma about the Minkowski functional only for the next proposition.

**Lemma 3.2.5.** *Let  $B$  be an absolutely convex, radially compact and absorbent subset of a real vector space  $E$ . Then for all  $\alpha \in \mathbb{R}_{>0}$  and  $y \in E$*

$$\alpha B \subseteq (\|y\|_B + \alpha)B + y.$$

*Proof.* If  $x \in \alpha B$ , then  $\|x\|_B \leq \alpha$ . This implies  $\|x - y\|_B \leq \|x\|_B + \|y\|_B = \|y\|_B + \alpha$ . The radial compactness implies  $x - y \in (\|y\|_B + \alpha)B$  (Lemma 0.1.7), and therefore  $x \in (\|y\|_B + \alpha)B + y$ .  $\square$

In the following proposition we show how to redefine the topology on any locally convex space with a compact barrel.

**Proposition 3.2.6.** *Let  $(E, \mathcal{T}, B)$  be a locally convex space with  $B$  a compact barrel. Then:*

- (i)  $\mathcal{T}_b = \{U \subseteq E \mid \forall \alpha \in \mathbb{R}_{>0}. \exists U_\alpha \in \mathcal{T}. U \cap \alpha B = U_\alpha \cap \alpha B\}$  is a topology.
- (ii)  $\mathcal{T}_b$  is the finest topology agreeing with  $\mathcal{T}$  on  $\alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ .
- (iii)  $\mathcal{T}_b$  is Hausdorff, and using  $\mathcal{N}_x$  to refer to the neighbourhood filter at  $x$ , we have  $\mathcal{N}_x = \mathcal{N}_0 + x$ .

*Proof.*

- (i) We see that  $\emptyset \in \mathcal{T}_b$  and  $E \in \mathcal{T}_b$  because they are both in  $\mathcal{T}$  and so  $\emptyset \cap \alpha B = \emptyset \cap \alpha B$  and likewise for  $E$ .

Let  $U, V \in \mathcal{T}_b$ , with  $U_\alpha$  and  $V_\alpha$  defined as expected. We see that

$$\begin{aligned} (U \cap V) \cap \alpha B &= (U \cap \alpha B) \cap (V \cap \alpha B) = (U_\alpha \cap \alpha B) \cap (V_\alpha \cap \alpha B) \\ &= (U_\alpha \cap V_\alpha) \cap \alpha B, \end{aligned}$$

so  $U \cap V \in \mathcal{T}_b$ .

If we let  $(U_i)_{i \in I}$  be a family of sets in  $\mathcal{T}_b$ , with  $U_{i,\alpha}$  the corresponding families of elements of  $\mathcal{T}$ , then

$$\left( \bigcup_{i \in I} U_i \right) \cap \alpha B = \bigcup_{i \in I} (U_i \cap \alpha B) = \bigcup_{i \in I} (U_{i,\alpha} \cap \alpha B) = \left( \bigcup_{i \in I} U_{i,\alpha} \right) \cap \alpha B,$$

so  $\mathcal{T}_b$  is closed under unions too, and is therefore a topology.

- (ii) There are two parts, the first is showing  $\mathcal{T}_b$  is finer than any topology agreeing with  $\mathcal{T}$  on each set  $\alpha B$  where  $\alpha \in \mathbb{R}_{>0}$ . The second is to show that  $\mathcal{T}_b$  agrees with  $\mathcal{T}$  on each  $\alpha B$ .

Let  $\mathcal{S}$  be a topology on  $E$  that agrees with  $\mathcal{T}$  on each set  $\alpha B$ . We need to show that  $\mathcal{S} \subseteq \mathcal{T}_b$ . Let  $U \in \mathcal{S}$ . Since  $\mathcal{S}$  agrees with  $\mathcal{T}$  on  $\alpha B$ , there is some  $U_\alpha \in \mathcal{T}$  such that  $U_\alpha \cap \alpha B = U \cap \alpha B$ . We have therefore shown  $U \in \mathcal{T}_b$ . It follows that  $\mathcal{T}_b$  is also finer than  $\mathcal{T}$ .

We must now show that  $\mathcal{T}_b$  agrees with  $\mathcal{T}$  on each  $\alpha B$ . Since  $\mathcal{T}_b$  is finer than  $\mathcal{T}$ ,  $\mathcal{T}_b|_{\alpha B}$  is finer than  $\mathcal{T}|_{\alpha B}$ . To prove the above, we must show that  $\mathcal{T}|_{\alpha B}$  is finer than  $\mathcal{T}_b|_{\alpha B}$ . If  $U' \in \mathcal{T}_b|_{\alpha B}$ , then  $U' = U \cap \alpha B$  for some  $U \in \mathcal{T}_b$ . Then  $U \cap \alpha B = U_\alpha \cap \alpha B$  for some  $U_\alpha \in \mathcal{T}$ , so  $U' \in \mathcal{T}|_{\alpha B}$ .

- (iii) Since we showed in (ii) that  $\mathcal{T}_b$  is finer than  $\mathcal{T}$ , it is Hausdorff because  $\mathcal{T}$  is. To prove the rest of the statement, we will first show  $\mathcal{N}_x + y \subseteq \mathcal{N}_{x+y}$  in  $\mathcal{T}_b$ . If  $N \in \mathcal{N}_x$ , we have that there is a  $U \in \mathcal{T}_b$  such that  $x \in U \subseteq N$ . We see that  $x + y \in U + y \subseteq N + y$ , so we have proven the inclusion of neighbourhoods if we can show that  $U + y \in \mathcal{T}_b$ . We make the definition  $(U + y)_\alpha = U_{\alpha + \|y\|_B} + y$ . Then we have

$$\begin{aligned} U \cap (\alpha + \|y\|_B)B &= U_{\alpha + \|y\|_B} \cap (\alpha + \|y\|_B)B \\ &\Rightarrow (U + y) \cap ((\alpha + \|y\|_B)B + y) = (U + y)_\alpha \cap (\alpha B + y) \\ &\Rightarrow (U + y) \cap \alpha B = (U + y)_\alpha \cap \alpha B, \end{aligned}$$

by Lemma 3.2.5. This shows  $U + y \in \mathcal{T}_b$ .

Now that we have established that  $\mathcal{N}_x + y \subseteq \mathcal{N}_{x+y}$ , we have  $\mathcal{N}_0 + x \subseteq \mathcal{N}_x$ , and  $\mathcal{N}_x + -x \subseteq \mathcal{N}_0$ , and by adding  $x$  we get  $\mathcal{N}_x \subseteq \mathcal{N}_0 + x$ .  $\square$

We now show how  $\mathcal{T}$  and  $\mathcal{T}_b$  relate to the weak topology,  $\sigma(E, E^*)$ .

**Lemma 3.2.7.** *Let  $(E, \mathcal{T}, B)$  be a locally convex space with compact barrel  $B$ . Then  $\mathcal{T}$  and  $\sigma(E, E^*)$  agree on each set  $\alpha B$  for  $\alpha \in \mathbb{R}_{>0}$ , and so the topology  $\mathcal{T}_b = \sigma(E, E^*)_b$ .*

*Proof.* Since  $\alpha \cdot -$  is continuous,  $\alpha B$  is compact for all  $\alpha \in \mathbb{R}_{>0}$ . We also have that by definition  $\sigma(E, E^*)$  is coarser than  $\mathcal{T}$ , and therefore the identity map  $\text{id} : (E, \mathcal{T}) \rightarrow (E, \sigma(E, E^*))$  is continuous. Therefore  $\alpha B$  is compact in  $\sigma(E, E^*)$ , and it is also Hausdorff because  $\sigma(E, E^*)$  is. A consequence of this is that  $\text{id} : (\alpha B, \mathcal{T}|_B) \rightarrow (\alpha B, \sigma(E, E^*)|_B)$  is a continuous bijection of compact Hausdorff spaces, and therefore a homeomorphism. Proposition 3.2.6 (ii) then implies that  $\mathcal{T}_b = \sigma(E, E^*)_b$ .  $\square$

We can conclude from the above that the topologies admitting a compact barrel are quite restricted in how they can behave.

The following definition and proposition draw on [28, V.5.4-5], which was not sufficiently general for our purposes.

If we have  $(E, \mathcal{T}, B)$ , a locally convex space with compact barrel  $B$ , and  $(\phi_i)_{i \in \mathbb{N}}$  is a sequence in  $E^*$  converging to 0 with respect to  $\|\cdot\|_{B^\circ}$ . Define

$$N_{(\phi_i)} = \{x \in E \mid \forall i \in \mathbb{N}. |\phi_i(x)| < 1\}$$

**Proposition 3.2.8.** *Given  $(E, \mathcal{T}, B)$ , the sets of the form  $N_{(\phi_i)}$  form a base for the neighbourhood filter of 0 in  $(E, \mathcal{T}_b)$ .*

*Proof.* We use Lemma 3.2.7 to reduce the problem to showing that the family of sets  $N_{(\phi_i)}$  is a neighbourhood base for 0 in  $(E, \sigma(E, E^*)_b)$ .

- $\{N_{(\phi_i)} \mid \phi_i \rightarrow 0 \text{ in } (E^*, \|\cdot\|_{B^o})\}$  is a filter base:

Recall the definition of a filter base from [16, I.6.3]: A set  $B$  of subsets of  $E$  is a filter base if the intersection of two sets from  $B$  contains a set from  $B$  and  $B$  is nonempty and does not contain the empty set. To show the first property, we need to prove one thing first. Let  $(\phi_i)_{i \in \mathbb{N}}$  and  $(\phi'_i)_{i \in \mathbb{N}}$  be sequences converging to zero in  $E^*$ . Define

$$\psi_{2i} = \phi_i \quad \psi_{2i+1} = \phi'_i.$$

Then we show that  $\psi_i \rightarrow 0$ . Let  $\epsilon \in \mathbb{R}_{>0}$ . There exist  $n, n' \in \mathbb{N}$  such that for all  $i \geq n$   $\|\phi_i\| < \epsilon$  and for all  $i \geq n'$ ,  $\|\phi'_i\| < \epsilon$ . Define  $m = 2 \max\{n, n'\} + 1$ . If  $i \geq m$  and is odd, then  $\|\psi_i\| = \|\phi_{\frac{i-1}{2}}\| < \epsilon$  because  $\frac{i-1}{2} \geq n'$ . Likewise, if  $i$  is even,  $\|\psi_i\| = \|\phi_{\frac{i}{2}}\| < \epsilon$  because  $\frac{i}{2} \geq n$ . Therefore  $N_{(\psi_i)}$  fits the definition. Now

$$\begin{aligned} N_{(\phi_i)} \cap N_{(\phi'_i)} &= \{x \in E \mid \forall i \in \mathbb{N}. |\phi_i(x)| < 1 \text{ and } |\phi'_i(x)| < 1\} \\ &= \{x \in E \mid \forall i \in \mathbb{N}. |\psi_{2i}(x)| < 1 \text{ and } |\psi_{2i+1}(x)| < 1\} \\ &= \{x \in E \mid \forall i \in \mathbb{N}. |\psi_i(x)| < 1\} \\ &= N_{(\psi_i)}. \end{aligned}$$

We have therefore verified the first property. To show that there always exists a set of the form  $N_{(\phi_i)}$ , we can take  $\phi_i = 0$  for all  $i \in \mathbb{N}$ . Then  $N_{(\phi_i)} = E$ . To show that all the  $N_{(\phi_i)}$  are nonempty, we observe that however  $(\phi_i)$  is defined, we always have  $|\phi(0)| = 0 < 1$ , so  $0 \in N_{(\phi_i)}$  for all  $(\phi_i)$ .

- $N_{(\phi_i)}$  is a  $\sigma(E, E^*)_b$ -open neighbourhood of 0:

We just showed that  $N_{(\phi_i)}$  always contains 0, so we only need to show that it is  $\sigma(E, E^*)$ -open. Recall from the definition of  $\sigma(E, E^*)$  that  $(N_\phi)_{\phi \in E^*}$  is a subbase of open 0-neighbourhoods (see (0.2)):

$$N_\phi = \{x \in E \mid |\phi(x)| < 1\}.$$

Observe that  $N_{(\phi_i)} = \bigcap_{i \in \mathbb{N}} N_{\phi_i}$ . We can prove  $N_{(\phi_i)}$  is  $\sigma(E, E^*)_b$ -open by finding, for each  $\alpha \in \mathbb{R}_{>0}$ , a finite set  $\{\phi_1, \dots, \phi_n\} \subseteq E^*$  such that  $N_{(\phi_i)} \cap \alpha B = \bigcap_{j=1}^n N_{\phi_j} \cap \alpha B$ .

To do this, we start with the fact that for all  $\epsilon \in \mathbb{R}_{>0}$ , there is an  $n_\epsilon \in \mathbb{N}$  such that for all  $i \geq n_\epsilon$ ,  $\|\phi_i\|_{B^\circ} < \epsilon$ . If  $i \geq n_{(2\alpha)^{-1}}$ , then  $\|\phi_i\|_{B^\circ} < (2\alpha)^{-1}$ , so since  $B^\circ$  is the closed unit ball (Lemma 3.2.2),  $\phi_i \in (2\alpha)^{-1}B^\circ = \frac{1}{2}\alpha B^\circ$  (Lemma 0.3.11 (ii)). So we have that for all  $i \geq n_{(2\alpha)^{-1}}$  and all  $x \in \alpha B$

$$|2\phi_i(x)| \leq 1 \Leftrightarrow |\phi_i(x)| \leq \frac{1}{2} \Rightarrow |\phi_i(x)| < 1 \Leftrightarrow x \in N_{\phi_i}.$$

So  $\alpha B \subseteq N_{\phi_i}$  for  $i \geq n_{(2\alpha)^{-1}}$ .

Let  $m = n_{(2\alpha)^{-1}}$ , and

$$\begin{aligned} N_{(\phi_i)} \cap \alpha B &= \bigcap_{i=1}^{\infty} N_{\phi_i} \cap \alpha B = \left( \bigcap_{i=1}^m N_{\phi_i} \alpha B \right) \cap \bigcap_{i=m+1}^{\infty} \alpha B \\ &= \left( \bigcap_{i=1}^m N_{\phi_i} \right) \cap \alpha B. \end{aligned}$$

Now,  $\bigcap_{i=1}^m N_{\phi_i}$  is  $\sigma(E, E^*)$ -open, and since this can be done for all  $\alpha \in \mathbb{R}_{>0}$ , we have shown  $N_{(\phi_i)}$  is  $\sigma(E, E^*)_b$ -open.

- The sets  $N_{(\phi_i)}$  generate the neighbourhood filter for 0, *i.e.* for each  $\sigma(E, E^*)_b$  0-neighbourhood  $N$ , there is a  $(\phi_i)$  such that  $\phi_i \rightarrow 0$  in  $(E^*, \|\cdot\|_{B^\circ})$  such that  $N_{(\phi_i)} \subseteq N$ :

Let  $N$  be a  $\sigma(E, E^*)_b$  0-neighbourhood, and  $U \subseteq N$  its interior, which is necessarily an open 0-neighbourhood. We construct  $(\phi_i)_{i \in \mathbb{N}}$  inductively. We define two countable families of finite subsets of  $E^*$ , which we call  $(X_i)_{i \in \mathbb{N}}$  and  $(Y_i)_{i \in \mathbb{N}}$ , such that  $X_i^{|\cdot|} \cap iB \subseteq U$ ,  $X_{i+1} = X_i \cup Y_{i+1}$ , and for  $i > 1$ ,  $Y_{i+1} \subseteq \frac{1}{i}B^{|\cdot|}$ .

We first observe that  $U \cap B = U_1 \cap B$  for some  $U_1$  that is  $\sigma(E, E^*)_b$ -open. We therefore have that there are  $\phi_1, \dots, \phi_n \in E^*$  such that  $\bigcap_{i=1}^n N_{\phi_i} \subseteq U_1$ . We define  $Y_1 = X_1 = \{2\phi_1, \dots, 2\phi_n\}$ . We then observe that

$$\begin{aligned} X_1^{|\cdot|} &= \{x \in X \mid \forall i \in \{1, \dots, n\}. |2\phi_i(x)| \leq 1\} \\ &= \left\{ x \in X \mid \forall i \in \{1, \dots, n\}. |\phi_i(x)| \leq \frac{1}{2} \right\} \\ &\subseteq \{x \in X \mid \forall i \in \{1, \dots, n\}. |\phi_i(x)| < 1\} \\ &= \bigcap_{i=1}^n N_{\phi_i} \subseteq U_1, \end{aligned}$$



so  $X_1^{|\circ|} \cap B = U_1 \cap B \subseteq U$ .

The inductive step proceeds as follows. Assume that there is a subset  $X_i \subseteq E^*$  such that  $X_i^{|\circ|} \cap iB \subseteq U$ . We only use this part of the inductive hypothesis. We show that there exists a finite  $Y_{i+1} \subseteq \frac{1}{i}B^{|\circ|}$  such that

$$(X_i \cup Y_{i+1})^{|\circ|} \cap (i+1)B \subseteq U$$

by contradiction. Assume for a contradiction that if  $Y \subseteq \frac{1}{i}B^{|\circ|}$  is finite, then  $(X_i \cup Y)^{|\circ|} \cap (i+1)B \not\subseteq U$ . We define  $\mathcal{F}$  to be the set of all subsets of  $E^*$  of the form  $(X_i \cup Y)^{|\circ|} \cap (i+1)B \cap (E \setminus U)$ , with  $Y \subseteq \frac{1}{i}B^{|\circ|}$ . By the assumption,  $\mathcal{F}$  consists of non-empty sets. Now,  $(X_i \cup Y)^{|\circ|}$  is  $\sigma(E, E^*)$ -closed because it is an absolute polar, and  $(i+1)B \cap E \setminus U$  is  $\sigma(E, E^*)$ -closed because  $U$  is  $\sigma(E, E^*)_b$ -open. So  $\mathcal{F}$  consists of closed subsets of  $(i+1)B$ . If  $Y$  and  $Y'$  are sets such that

$$\begin{aligned} (X_i \cup Y)^{|\circ|} \cap (i+1)B \cap (E \setminus U) &\in \mathcal{F} \\ (X_i \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) &\in \mathcal{F}, \end{aligned}$$

then we have

$$\begin{aligned} &(X_i \cup Y)^{|\circ|} \cap (i+1)B \cap (E \setminus U) \cap (X_i \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) \\ &= (X_i \cup Y)^{|\circ|} \cap (X_i \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) \\ &= (X_i \cup Y \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) \quad \text{Lemma 0.3.11 (i).} \end{aligned}$$

The set  $Y \cup Y_i$  is also a finite subset of  $\frac{1}{i}B^{|\circ|}$ , so we have shown that  $\mathcal{F}$  is closed under finite intersections, and any finite intersection of sets in  $\mathcal{F}$  is non-empty. By the intersection formulation of compactness [16, I.9.1 (C'')],  $\bigcap \mathcal{F}$  is not empty. Let  $x \in \bigcap \mathcal{F}$ . By Lemma 0.3.11(iii),  $(X_i \cup Y)^{|\circ|} \subseteq X_i^{|\circ|}$  for any set  $Y$ , so  $x \subseteq X_i^{|\circ|} \cap (i+1)B \cap (E \setminus U)$ . We also have that for all  $\phi \in \frac{1}{i}B^{|\circ|}$ ,  $x \in \{\phi\}^{|\circ|}$ , which is to say,  $|\phi(x)| \leq 1$ . Therefore

$$\begin{aligned} x &\in \left\{ x' \in X \mid \forall \phi \in \frac{1}{i}B^{|\circ|}, |\phi(x)| \leq 1 \right\} \\ &= \left( \frac{1}{i}B^{|\circ|} \right)^{|\circ|} \\ &= (iB)^{|\circ||\circ|} \quad \text{Lemma 0.3.11 (ii)} \\ &= iB \quad \text{Corollary 0.3.12,} \end{aligned}$$

so in fact,  $x \in X_i^{|\circ|} \cap iB \cap E \setminus U$ , which contradicts the inductive hypothesis that  $X_i^{|\circ|} \cap iB \subseteq U$ .

We then define  $Y_i$  to be a finite subset of  $\frac{1}{i}B^{|\circ|}$  that we have just shown to exist, and define  $X_{i+1} = X_i + Y_{i+1}$ . We now have  $X_{i+1}^{|\circ|} \cap (i+1)B \subseteq U$ , as required. This finishes the inductive construction.

Define  $(\phi_i)_{i \in \mathbb{N}}$  to enumerate the elements of the  $Y_i$  in increasing order of  $i$ .

To see that  $\phi_i \rightarrow 0$  for  $\|\cdot\|_{B^{|\circ|}}$ , let  $\epsilon \in \mathbb{R}_{>0}$ . There is a smallest  $m$  such that  $\frac{1}{m} < \epsilon$ . For all  $i \geq m$ , any  $\phi \in Y_i$  is in  $\frac{1}{m}B^{|\circ|}$ . We may now define  $n = \sum_{j=1}^m |Y_j| + 1$ . Then if  $i \geq n$  we have  $\phi_i \in \frac{1}{m}B^{|\circ|}$ , so  $\|\phi_i\|_{B^{|\circ|}} \leq \frac{1}{m} < \epsilon$ , proving convergence to 0.

All that remains is to show that  $N_{(\phi_i)} \subseteq U$ , as  $U \subseteq N$ . From the definitions, we have  $N_{(\phi_i)} \subseteq (\phi_i)^{|\circ|}$ , as it is a change from a strict inequality to one that is not strict. It therefore suffices to show that  $(\phi_i)^{|\circ|} \subseteq U$ . We have

$$(\phi_i)^{|\circ|} = \left( \bigcup_{i=1}^{\infty} Y_i \right)^{|\circ|} = \left( \bigcup_{i=1}^{\infty} X_i \right)^{|\circ|} = \bigcap_{i=1}^{\infty} X_i^{|\circ|},$$

by Lemma 0.3.11, and we constructed the  $(X_i)$  so that  $X_i^{|\circ|} \cap iB \subseteq U$ . So for all  $i \in \mathbb{N}$

$$\left( \bigcup_{j=1}^{\infty} X_j^{|\circ|} \right) \cap iB \subseteq X_i^{|\circ|} \cap iB \subseteq U.$$

Therefore

$$\begin{aligned} (\phi_i)^{|\circ|} &= \bigcap_{i=1}^{\infty} X_i^{|\circ|} = \left( \bigcap_{i=1}^{\infty} X_i^{|\circ|} \right) \cap E = \left( \bigcap_{i=1}^{\infty} X_i^{|\circ|} \right) \cap \left( \bigcup_{j=1}^{\infty} jB \right) \\ &= \bigcup_{j=1}^{\infty} \left( \bigcap_{i=1}^{\infty} X_i^{|\circ|} \right) \cap jB. \end{aligned}$$

As each part of the big union is a subset of  $U$ , the union is too, so we have shown  $(\phi_i)^{|\circ|} \subseteq U$ .  $\square$

**Proposition 3.2.9.** *If  $(E, \mathcal{T}, B)$  is a locally convex space with compact barrel  $B$ , then  $\mathcal{T}_b$  is locally convex and  $(E, \mathcal{T}_b, B)$  is a Smith space.*

*Proof.* We prove this by showing that the filter base

$$\mathcal{N} = \{N_{(\phi_i)} \mid (\phi_i) \text{ converges to } 0 \text{ in } (E^*, \|\cdot\|_{B|\circ|})\}$$

defines a locally convex topology  $\mathcal{S}$  using [15, II.4.1 Proposition 1].<sup>1</sup> Then Proposition 3.2.8 implies  $(E, \mathcal{T}_b)$  has the same neighbourhood filter at zero as  $\mathcal{S}$ , and therefore has the same neighbourhood filter at every point by Proposition 3.2.6 (iii).

We must therefore show that each  $N_{(\phi_i)} \in \mathcal{N}$  is absorbent, absolutely convex and that  $\alpha N_{(\phi_i)} \in \mathcal{N}$  for all  $\alpha \in \mathbb{R}_{>0}$ .

- Each  $N_{(\phi_i)} \in \mathcal{N}$  is absorbent:

We showed in Proposition 3.2.8 that  $N_{(\phi_i)} \supseteq N_{(\phi_i)} \cap B = (\bigcap_{i=1}^n N_{\phi_i}) \cap B$  for some  $n \in \mathbb{N}$ . It is not hard to prove directly that  $N_{\phi_i}$  is absorbent, but we can also deduce this from Lemma 0.1.4. By assumption,  $B$  is absorbent, so by Lemma 0.1.3  $\bigcap_{i=1}^n N_{\phi_i} \cap B$  is absorbent and so  $N_{(\phi_i)}$  is absorbent.

- $N_{(\phi_i)}$  is absolutely convex:

Let  $\sum_{i \in I} \alpha_i x_i$  be a finite absolutely convex combination of elements of  $N_{(\phi_i)}$ . Then for all  $i \in I$  and  $j \in \mathbb{N}$ ,  $|\phi_j(x_i)| < 1$ . Therefore

$$\left| \phi_j \left( \sum_{i \in I} \alpha_i x_i \right) \right| = \left| \sum_{i \in I} \alpha_i \phi_j(x_i) \right| \leq \sum_{i \in I} |\alpha_i| \cdot |\phi_j(x_i)| < \sum_{i \in I} |\alpha_i| \leq 1,$$

so  $\sum_{i \in I} \alpha_i x_i \in N_{(\phi_i)}$ .

- $N_{(\phi_i)} \in \mathcal{N}$  implies  $\alpha N_{(\phi_i)} \in \mathcal{N}$  for all  $\alpha \in \mathbb{R}_{>0}$ :

We show this by proving that  $\alpha N_{(\phi_i)} = N_{(\alpha^{-1}\phi_i)}$ , analogously to Lemma 0.3.11 (ii), and then continuity of scalar multiplication on  $(E^*, \|\cdot\|_{B^\circ})$  shows that  $(\alpha^{-1}\phi_i)_{i \in \mathbb{N}}$  converges to 0, so  $N_{(\alpha^{-1}\phi_i)} \in \mathcal{N}$ .

$$\begin{aligned} x \in \alpha N_{(\phi_i)} &\Leftrightarrow \alpha^{-1}x \in N_{(\phi_i)} \Leftrightarrow \forall i \in \mathbb{N}. |\phi_i(\alpha^{-1}x)| < 1 \\ &\Leftrightarrow \forall i \in \mathbb{N}. |\alpha^{-1}\phi_i(x)| < 1 \Leftrightarrow x \in N_{(\alpha^{-1}\phi_i)}. \end{aligned}$$

So the two sets are the same.

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<sup>1</sup>See [16, I.1.2 Proposition 2 and III.1.2 Proposition 1], then [15, I.1.5 Proposition 4] for the whole story.

Therefore  $\mathcal{T}_b$  is a locally convex topology. We know that  $\mathcal{T}_b$  agrees with  $\mathcal{T}$  on  $\alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ , and that it is the finest such topology. Therefore it is the finest topology agreeing with  $\mathcal{T}_b$  on  $\alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ , so  $(E, \mathcal{T}_b, B)$  is a Smith space, by our definition.  $\square$

For reasons that should be clear, we call  $(E, \mathcal{T}_b, B)$  the *Smithification* of  $(E, \mathcal{T}, B)$ .

**Lemma 3.2.10.** *Let  $(E, \mathcal{T}, B)$  be a Smith space.*

- (i) *If  $U \subseteq E$  is a set such that  $U \cap \alpha B$  is  $\mathcal{T}$ -open in  $\alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ , then  $U$  is  $\mathcal{T}$ -open.*
- (ii) *If  $C \subseteq E$  is a set such that  $C \cap \alpha B$  is  $\mathcal{T}$ -closed in  $\alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ , then  $C$  is  $\mathcal{T}$ -closed.*

*Proof.*

- (i) We can take  $\mathcal{T}_b = \{U \subseteq E \mid \forall \alpha \in \mathbb{R}_{>0}. \exists U_\alpha \in \mathcal{T}. U \cap \alpha B = U_\alpha \cap \alpha B\}$  as in Proposition 3.2.6. By part (ii) of that proposition,  $\mathcal{T}_b$  is the finest topology agreeing with  $\mathcal{T}$  on  $\alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ , so by our assumption that  $(E, \mathcal{T}, B)$  is Smith,  $\mathcal{T}_b = \mathcal{T}$ . Since  $U \in \mathcal{T}_b$ , we have that  $U \in \mathcal{T}$ .
- (ii) We deduce this from part (i) as follows. We know that  $C \cap \alpha B = C_\alpha \cap \alpha B$  for all  $\alpha \in \mathbb{R}_{>0}$ , where  $C_\alpha$  is a  $\mathcal{T}$ -closed set. Now

$$\begin{aligned} (E \setminus C) \cap \alpha B &= \alpha B \setminus C = \alpha B \setminus (\alpha B \cap C) = \alpha B \setminus (\alpha B \cap C_\alpha) \\ &= (E \setminus C_\alpha) \cap \alpha B. \end{aligned}$$

Sine  $E \setminus C_\alpha$  is  $\mathcal{T}$ -open for all  $\alpha \in \mathbb{R}_{>0}$ , we deduce from (i) that  $E \setminus C$  is  $\mathcal{T}$ -open, and therefore  $C$  is  $\mathcal{T}$ -closed.  $\square$

We now introduce a notation. If  $(E, \mathcal{T}, B)$  a locally convex space with compact barrel  $B$ , such as a Smith space, we define  $(E, \mathcal{T}, B)^\beta$ , or  $E^\beta$  where no confusion is possible, to be  $E^*$  with the topology defined by  $\|-\|_{B^\circ}$ . We use the letter  $\beta$  because it is associated to strong topologies, although we have not yet shown that for a Smith space  $(E, \mathcal{T}, B)$ ,  $E^\beta$  is the dual space with the strong dual topology. The choice of the letter  $\beta$  comes from the relationship between the strong dual topology and uniform convergence on *bounded* sets. We have already shown in Lemma 3.2.2 that  $E^\beta$  is always a normed space. We aim to show that in the case that  $E$  is a Smith space,  $E^\beta$  is in fact a Banach space. To do this we first need a lemma.

**Lemma 3.2.11.** *Let  $(E, \mathcal{T}, B)$  be a locally convex space with a compact barrel. The family*

$$\mathfrak{S}_B = \{S \subseteq \alpha B \mid \alpha \in \mathbb{R}_{>0}\}$$

*is a saturated family (in the sense of [110, p. 81]) that covers  $E$  and consists of sets that are bounded.*

*Proof.* There are three conditions to check for  $\mathfrak{S}_B$  to be a saturated family ([110, p.81]).

- (i)  $\mathfrak{S}_B$  contains all subsets of its elements: Trivially implied by the definition.
- (ii)  $\mathfrak{S}_B$  contains all scalar multiples of its elements: Let  $S \subseteq \alpha B$ . We then have that  $\beta S \subseteq \alpha\beta B$ , so  $\beta S \in \mathfrak{S}_B$ .
- (iii)  $\mathfrak{S}_B$  contains the closed absolutely convex hull of all finite unions of its elements: Let  $(S_i)_{i \in I}$  be a finite family of elements of  $\mathfrak{S}_B$ , with  $(\alpha_i)_{i \in I}$  being defined such that  $S_i \subseteq \alpha_i B$ . Let  $j$  be the index of  $\max_{i \in I} \alpha_i$ . Then for all  $i \in I$ ,  $S_i \subseteq \alpha_j B$ , and so  $\bigcup_{i \in I} S_i \subseteq \alpha_j B$ . Then  $\alpha_j B$  is closed and absolutely convex, so the closed absolutely convex hull of  $\bigcup_{i \in I} S_i$  is also a subset of  $\alpha_j B$ , hence an element of  $\mathfrak{S}_B$ .

We see that  $\alpha B$  covers  $E$  because  $B$  is absorbent. Each  $S \in \mathfrak{S}_B$  is bounded because compact sets are bounded (Lemma 0.1.14) and any subset of a bounded set is bounded.  $\square$

For any saturated family of sets on a locally convex space  $E$ , we can define a locally convex topology on  $E^*$  by using the polars of sets from that family as a base for 0-neighbourhoods ([110, III.3.2 Corollary] with  $F = \mathbb{R}$ ).

**Lemma 3.2.12.** *Given a locally convex space with compact barrel  $(E, \mathcal{T}, B)$ , the  $\mathfrak{S}_B$ -topology on  $E^*$  is the  $\|\cdot\|_{B^\circ}$  topology.*

*Proof.* Suppose  $N \subseteq E$  is a  $\mathfrak{S}_B$  0-neighbourhood, i.e. there exists an  $S \in \mathfrak{S}_B$  such that  $S^\circ \subseteq N$ . Then  $S \subseteq \alpha B$  for some  $\alpha \in \mathbb{R}_{>0}$ , so by Lemma 0.3.11(ii) and (iii),  $\alpha^{-1}B^\circ = (\alpha B)^\circ \subseteq S^\circ \subseteq N$ , so  $N$  is a  $\|\cdot\|_{B^\circ}$  0-neighbourhood.

In the other direction, let  $N$  be a  $\|\cdot\|_{B^\circ}$  0-neighbourhood, i.e. there exists some  $\alpha \in \mathbb{R}_{>0}$  such that  $\alpha B^\circ \subseteq N$ . Then by Lemma 0.3.11 (ii),  $(\alpha^{-1}B)^\circ \subseteq N$ , so  $N$  is a  $\mathfrak{S}_B$  0-neighbourhood.  $\square$

**Proposition 3.2.13.** *Let  $(E, \mathcal{T}, B)$  be a locally convex space with  $B$  a compact barrel. Then  $(E, \mathcal{T}_b, B)^\beta$  is isomorphic to the completion of  $(E, \mathcal{T}, B)^\beta$ . In particular, if  $(E, \mathcal{T}, B)$  is a Smith space,  $E^\beta$  is a Banach space.*

*Proof.* We use Grothendieck's completeness theorem ([110, Theorem IV.6.2]). This states that  $E^*$  is complete in the  $\mathfrak{S}_B$  topology iff every  $\phi : E \rightarrow \mathbb{R}$  which is continuous when restricted to any  $S \in \mathfrak{S}_B$  is continuous. We want to use this to show that  $(E, \mathcal{T}, B)^\beta$  is complete whenever  $(E, \mathcal{T}, B)$  is a Smith space, identifying the norm topology with the  $\mathfrak{S}_B$  topology by Lemma 3.2.12.

Let  $(E, \mathcal{T}, B)$  be a Smith space and let  $\phi : E \rightarrow \mathbb{R}$  be a linear map such that for all  $S \in \mathfrak{S}_B$ ,  $\phi|_S$  is continuous, where  $E$  has the subspace topology from  $\mathcal{T}$ . Then *a fortiori* we have that  $\phi|_{\alpha B}$  is continuous, and so if  $V \subseteq \mathbb{R}$  is an open set, there exists  $U_\alpha \in \mathbb{R}$  such that  $f^{-1}(V) \cap \alpha B = U_\alpha \cap \alpha B$ . By Lemma 3.2.10,  $f^{-1}(V) \in \mathcal{T}$ , so  $\phi$  is continuous. We have therefore shown  $E^\beta$  is complete for any Smith space  $(E, \mathcal{T}, B)$ , and so if  $(F, \mathcal{S}, C)$  is a locally convex space with compact barrel  $B$ , then  $(F, \mathcal{S}_b, C)^\beta$  is complete by Proposition 3.2.9.

It remains to show that if  $(E, \mathcal{T}, B)$  is a locally convex space with compact barrel  $B$ , that  $(E, \mathcal{T}_b, B)^\beta$  is isomorphic to the completion of  $(E, \mathcal{T}, B)$ . By a standard theorem ([16, II.3.7 Proposition 13]) this can be proven by showing that  $(E, \mathcal{T}, B)^\beta$  is dense in  $(E, \mathcal{T}_b, B)^\beta$  and  $(E, \mathcal{T}, B)^\beta$  has the subspace topology as a subset of  $(E, \mathcal{T}_b, B)^\beta$  (it is a subset because  $\mathcal{T}_b$  is finer than  $\mathcal{T}$ ).

We show that  $(E, \mathcal{T}, B)^\beta$  has the subspace topology as follows. We use  $B_\mathcal{T}^\circ$  to mean the polar of  $B$  in  $(E, \mathcal{T})^*$ , and  $B_{\mathcal{T}_b}^\circ$  to mean the polar of  $B$  in  $(E, \mathcal{T}_b)^*$ . The unit ball of  $\|\cdot\|_{B_\mathcal{T}^\circ}$  is  $B_\mathcal{T}^\circ$ . The unit ball of the subspace topology of  $\|\cdot\|_{B_{\mathcal{T}_b}^\circ}$  in  $(E, \mathcal{T})^*$  is  $B_{\mathcal{T}_b}^\circ \cap (E, \mathcal{T})^*$ . By Lemma 0.3.13 these are equal, so the two norms and the topologies they generate are equal on  $(E, \mathcal{T})^* = (E, \mathcal{T}, B)^\beta$ .

To show that  $(E, \mathcal{T})^*$  is dense in  $(E, \mathcal{T}_b)^*$ , we use [110, IV.6.2 Corollary 1]. This states that if  $(E, F, \langle \cdot, \cdot \rangle)$  is a duality and  $\mathfrak{S}$  is a saturated family of weakly bounded sets covering  $E$ , and  $F_1$  is the space of linear maps  $\phi : E \rightarrow \mathbb{R}$  whose restrictions to each  $S \in \mathfrak{S}$  are  $\sigma(E, F)$ -continuous, given the  $\mathfrak{S}$ -topology, then  $G_1$  is complete and  $G$ , embedded in it via the pairing, is dense in it.

Now,  $\mathfrak{S}_B$  is a saturated family of bounded sets covering  $E$  (Lemma 3.2.11) and every bounded set is weakly bounded because every weak 0-neighbourhood is a 0-neighbourhood in  $\mathcal{T}$ . We also have a duality  $(E, (E, \mathcal{T})^*)$ , by Proposition 0.3.1. Let  $\phi : E \rightarrow \mathbb{R}$  be a linear map. By Lemma 3.2.7,  $\phi$  is  $\sigma(E, (E, \mathcal{T})^*)$ -continuous when restricted to each  $S \in \mathfrak{S}$  iff it is  $\mathcal{T}$ -continuous when restricted to each  $S \in \mathfrak{S}$ . So by the argument in the second paragraph,  $\phi$  is  $\mathcal{T}_b$  continuous. If  $\phi$  is  $\mathcal{T}_b$  continuous, it is also  $\mathcal{T}$ -continuous when restricted to any  $S \in \mathfrak{S}$  (Proposition 3.2.6(ii)), so we have shown  $(E, \mathcal{T}_b)^* = (E, \mathcal{T})^*$ , so by [110, IV.6.2 Corollary 1],  $(E, \mathcal{T})^*$  is dense in  $(E, \mathcal{T}_b)^*$ , and therefore  $(E, \mathcal{T}_b, B)^\beta$  is the completion of  $(E, \mathcal{T}, B)^\beta$ .  $\square$

We can now prove the following fact about bounded sets in Smith spaces.

**Proposition 3.2.14.** *In any Smith space  $(E, \mathcal{T}, B)$ , if  $S \subseteq E$  is weakly bounded, then there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $S \subseteq \alpha B$ , and therefore  $S$  is bounded. Therefore  $\{\alpha B\}_{\alpha \in \mathbb{R}_{>0}}$  is a fundamental family for both weakly bounded sets and bounded sets. We also have that for each compact set  $C \subseteq E$  there is an  $\alpha \in \mathbb{R}_{>0}$  such that  $C \subseteq \alpha B$ .*

*Proof.* Let  $S \subseteq E$  be a weakly bounded (i.e.  $\sigma(E, E^*)$ -bounded) set. We first show that  $S^{|\cdot|}$  is a barrel, i.e. a set that is closed, absolutely convex and absorbent.

By Lemma 0.3.5,  $S^{|\cdot|}$  is absolutely convex and  $\sigma(E^*, E)$ -closed. The topology  $\sigma(E^*, E)$  is, by definition, the coarsest locally convex topology such that for all  $x \in E$ ,  $\text{ev}(x) : E^* \rightarrow \mathbb{R}$  is continuous. We proved in Lemma 3.2.4 that  $\text{ev}(x)$  was continuous for the  $\|\cdot\|_{B^\circ}$  norm for all  $x \in E$ , and therefore  $\sigma(E^*, E)$  is coarser than the  $\|\cdot\|_{B^\circ}$  topology, so  $S^{|\cdot|}$  is also closed in this topology. To show that  $S^{|\cdot|}$  is absorbent, let  $\phi \in E^*$ . Since  $S$  is weakly bounded, there exists an  $\alpha \in \mathbb{R}_{>0}$  such that  $S \subseteq \alpha N_\phi$ , where  $N_\phi$  is the open  $\sigma(E, E^*)$  0-neighbourhood as in equation (0.2). So we have

$$\begin{aligned} \forall x \in S. x \in \alpha N_\phi &\Leftrightarrow \forall x \in S. \alpha^{-1}x \in N_\phi \\ &\Leftrightarrow \forall x \in \alpha^{-1}S. x \in N_\phi \\ &\Leftrightarrow \forall x \in \alpha^{-1}S. |\phi(x)| < 1 \\ &\Rightarrow \forall x \in \alpha^{-1}S. |\phi(x)| \leq 1 \\ &\Leftrightarrow \phi \in (\alpha^{-1}S)^{|\cdot|} \\ &\Leftrightarrow \phi \in \alpha S^{|\cdot|}. \end{aligned}$$

We have now shown  $S^{|\cdot|}$  is a barrel.

Since  $E^\beta$  is a Banach space, it is barrelled by [110, II.7.1 Corollary], and so  $S^{|\cdot|}$  is a 0-neighbourhood. Therefore there exists an  $\alpha \in \mathbb{R}_{>0}$  such that  $\alpha B^{|\cdot|} \subseteq S^{|\cdot|}$  (recall that  $B^{|\cdot|} = B^\circ$  by Lemma 0.3.6 and  $S^\circ$  is the closed unit ball by Lemma 3.2.2). By Lemma 0.3.11

$$S^{|\cdot||\cdot|} \subseteq (\alpha B^{|\cdot|})^{|\cdot|} = \alpha^{-1} B^{|\cdot||\cdot|}.$$

We then have

$$S \subseteq S^{|\cdot||\cdot|} \subseteq \alpha^{-1} B^{|\cdot||\cdot|} = \alpha^{-1} B$$

by the absolute bipolar theorem (Corollary 0.3.12).

Since  $\alpha B$  is compact, it is  $\mathcal{T}$ -bounded, by Lemma 0.1.14. Therefore every weakly bounded set is  $\mathcal{T}$ -bounded, the other implication holding by definition.

Therefore a set  $S \subseteq E$  is (weakly) bounded iff it is a subset of some  $\alpha B$  for  $\alpha \in \mathbb{R}_{>0}$ . Since compact sets are bounded, we also have that every compact set is contained in some  $\alpha B$ .  $\square$

The preceding proof shows that, for  $E$  a Smith space,  $\mathfrak{S}_B$  is actually the family of weakly bounded sets in  $E$ , so  $E^\beta$  is in fact the strong dual, as defined in [110, IV.5], justifying our choice of Greek letter. We also see that the previous result and Lemma 3.2.10 imply that every Smith space is compactly generated (see [81, VII.8] and [69, page 230] for compactly generated spaces, and [2, Proposition 4.7] for Akbarov's proof of this).

We can prove continuity of maps from Smith spaces to topological vector spaces more easily using the following proposition.

**Proposition 3.2.15.** *Let  $(E, \mathcal{T}, B)$  be a Smith space,  $(F, \mathcal{S})$  a topological vector space, and  $f : E \rightarrow F$  a linear map such that  $f|_B$  is continuous. Then  $f$  is continuous.*

*Proof.* As  $f|_B$  is continuous, we have that for all  $V \in \mathcal{S}$ , there exists a  $U \in \mathcal{T}$  such that  $f^{-1}(V) \cap B = U \cap B$ . The first step in proving  $f$  is continuous is to prove that  $f|_{\alpha B}$  is continuous for all  $\alpha \in \mathbb{R}_{>0}$ . So let  $V \in \mathcal{S}$ . Since  $(F, \mathcal{S})$  is a topological vector space,  $\alpha^{-1}V \in \mathcal{S}$ . Therefore there is a  $U_0 \in \mathcal{T}$  such that  $f^{-1}(\alpha^{-1}V) \cap B = U_0 \cap B$ . So

$$\begin{aligned} \alpha(f^{-1}(\alpha^{-1}V) \cap B) &= \alpha(U_0 \cap B) \\ \Leftrightarrow f^{-1}(V) \cap \alpha B &= \alpha U_0 \cap \alpha B. \end{aligned}$$

Since  $\alpha U_0 \in \mathcal{T}$ , we have shown  $f|_{\alpha B}$  is continuous.

If we fix an open set  $V \in \mathcal{S}$ , and denote by  $U_\alpha$  an open set such that  $f^{-1}(V) \cap \alpha B = U_\alpha \cap \alpha B$ , which we proved to exist in the previous paragraph for each  $\alpha \in \mathbb{R}_{>0}$ , we can see that  $f^{-1}(V) \in \mathcal{T}$  by Lemma 3.2.10.  $\square$

**Corollary 3.2.16.** *Let  $(E, \mathcal{T}, B)$  be a Smith space,  $(F, \mathcal{S}, C)$  a locally convex space with compact barrel  $C$ , and  $f : E \rightarrow F$  a linear map such that there is an  $\alpha \in \mathbb{R}_{>0}$  such that  $f(B) \subseteq \alpha C$  and  $f|_B$  is continuous for  $\mathcal{S}$ . Then  $f$  is continuous from  $(E, \mathcal{T})$  to  $(F, \mathcal{S}_b)$ .*

*Proof.* We first show that  $f|_B$  is continuous from  $(B, \mathcal{T}|_B)$  to  $(F, \mathcal{S}_b)$ . Let  $V \in \mathcal{S}_b$ . This means, in particular, that there exists a set  $V_\alpha \in \mathcal{S}$  such that  $V_\alpha \cap \alpha C = V \cap \alpha C$ . Then

$$\begin{aligned} f^{-1}(V_\alpha \cap \alpha C) &= f^{-1}(V \cap \alpha C) \\ \Leftrightarrow f^{-1}(V_\alpha) \cap f^{-1}(\alpha C) &= f^{-1}(V) \cap f^{-1}(\alpha C) \\ \Leftrightarrow f^{-1}(V_\alpha) \cap B &= f^{-1}(V) \cap B, \end{aligned}$$



by the assumption that  $B \subseteq f^{-1}(\alpha C)$ . By the assumed continuity of  $f|_B$  for  $\mathcal{S}$ ,  $f^{-1}(V_\alpha) \cap B = U \cap B$  for some  $U \in \mathcal{T}$ , so this shows that  $f^{-1}(V) \cap B = U \cap B$  and therefore  $f|_B$  is continuous for  $\mathcal{S}_b$  as well. By Proposition 3.2.15,  $f$  is continuous from  $(E, \mathcal{T})$  to  $(F, \mathcal{S}_b)$ .  $\square$

If  $E$  is a normed space, by the Banach-Alaoglu theorem [28, V.4 Theorem 2]<sup>2</sup>  $\text{Ball}(E^*)$ , of the dual norm, is compact in the weak-\* (or  $\sigma(E^*, E)$ ) topology. Since  $\text{Ball}(E^*)$  is absolutely convex and absorbent,  $(E, \sigma(E^*, E)_b, B)$  is a Smith space (Proposition 3.2.9). We denote this by  $E^\sigma$ . This is known as the *bounded weak-\* topology* [28, Definition V.5.3, Corollary V.5.5] [8, Chapter 1, Theorem 2.2], usually restricted to the case that  $E$  is Banach. We use the letter  $\sigma$  as it is associated to weak topologies (probably from *schwach*).

We now consider the embedding in the double dual, in particular

$$\begin{aligned} \text{ev} : E &\rightarrow E^{\beta\sigma} \\ \text{ev}(x)(\phi) &= \phi(x), \end{aligned}$$

where  $(E, \mathcal{T}, B)$  is a Smith space.

**Proposition 3.2.17.** *If  $(E, \mathcal{T}, B)$  is a Smith space, the map  $\text{ev} : E \rightarrow E^{\beta\sigma}$  is a linear homeomorphism preserving the unit ball. Therefore every Smith space is isomorphic to the bounded weak-\* dual of a Banach space, which can be taken to be  $E^\beta$ .*

*Proof.* We first show that  $x \in E$  implies  $\text{ev}(x) \in E^{\beta\sigma}$ . The underlying space of  $E^{\beta\sigma}$  is  $(E^*, \|\cdot\|_{B^\circ})^*$ , so  $\text{ev}(x) \in E^{\beta\sigma}$  iff  $\text{ev}(x) : E^* \rightarrow \mathbb{R}$  is continuous with respect to  $\|\cdot\|_{B^\circ}$ . This follows from Lemma 3.2.4.

To show that  $\text{ev}$  is continuous, we first show it is continuous if  $E^{\beta\sigma}$  is given the  $\sigma(E^{\beta\sigma}, E^\beta)$  topology. A subbasis for open neighbourhoods is given by the family of sets

$$N_\phi = \{\Phi \in E^{\beta\sigma} \mid |\Phi(\phi)| < 1\}$$

where  $\phi \in E^\beta$  (see (0.2)). Because preimages preserve intersections, we only need to show that  $\text{ev}^{-1}(N_\phi)$  is open for all  $\phi \in E^\beta$ . So

$$\begin{aligned} \text{ev}^{-1}(N_\phi) &= \{x \in E \mid \text{ev}(x) \in N_\phi\} = \{x \in E \mid |\text{ev}(x)(\phi)| < \epsilon\} \\ &= \{x \in E \mid |\phi(x)| < \epsilon\} = \{x \in E \mid \phi(x) \in (-\epsilon, \epsilon)\} = \phi^{-1}((-\epsilon, \epsilon)), \end{aligned}$$

which is open because  $\phi$  is continuous.

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<sup>2</sup>Though the theorem is stated for Banach spaces in this reference, the proof does not use completeness.

We denote the unit ball of  $E^{\beta\sigma}$  by  $C$ . This is the polar of the unit ball of  $E^\beta$ , which is  $B^\circ$  (Lemma 3.2.2), but the polars are with respect to different pairings. We show that

$$B = \text{ev}^{-1}(C). \quad (3.4)$$

By definition

$$C = \{\Phi \in E^{\beta\sigma} \mid \forall \phi \in B^\circ. |\Phi(\phi)| \leq 1\},$$

so

$$\begin{aligned} \text{ev}^{-1}(C) &= \{x \in E \mid \text{ev}(x) \in C\} = \{x \in E \mid \forall \phi \in B^\circ. |\text{ev}(x)(\phi)| \leq 1\} \\ &= \{x \in E \mid \forall \phi \in B^\circ. |\phi(x)| \leq 1\} = B^{\circ\circ}, \end{aligned}$$

and  $B^{\circ\circ} = B$  by Corollary 0.3.10 as  $B$  is closed.

Now, let,  $U$  be an open subset of  $E^{\beta\sigma}$ , *i.e.*  $U \in \sigma(E^{\beta\sigma}, E^\beta)_b$ , which is to say that for all  $\alpha > 0$ , there exists  $U_\alpha \in \sigma(E^{\beta\sigma}, E^\beta)$  such that  $U \cap \alpha C = U_\alpha \cap \alpha C$ . Then

$$\begin{aligned} \text{ev}^{-1}(U) \cap \alpha B &= \text{ev}^{-1}(U) \cap \alpha \text{ev}^{-1}(C) & (3.4) \\ &= \text{ev}^{-1}(U \cap \alpha C) & \text{linearity} \\ &= \text{ev}^{-1}(U_\alpha \cap \alpha C) \\ &= \text{ev}^{-1}(U_\alpha) \cap \alpha B. \end{aligned}$$

We already showed that  $\text{ev}^{-1}(U_\alpha)$  is  $\mathcal{T}$ -open, so  $U$  is  $\mathcal{T}$ -open by Lemma 3.2.10, and we have shown that  $\text{ev}$  is continuous.

To see that  $\text{ev}$  is injective, suppose  $x, y \in E$  and  $\text{ev}(x) = \text{ev}(y)$ . Then for all  $\phi \in E^*$ , we have  $\text{ev}(x)(\phi) = \text{ev}(y)(\phi)$ , *i.e.*  $\phi(x) = \phi(y)$  and so  $\phi(x - y) = 0$ . Since the pairing between  $E$  and  $E^*$  is separating,  $x = y$  (Proposition 0.3.1).

To show that  $\text{ev}$  is surjective, we first show that  $\text{ev}(B) = C$ . We do this by showing first that  $\text{Ball}(E^\beta) = \text{ev}(B)^\circ$ , where the polar is with respect to the  $(E^\beta, E^{\beta\sigma})$  duality. We have

$$\text{ev}(B)^\circ = \{\phi \in E^\beta \mid \forall \Phi \in \text{ev}(B). |\Phi(\phi)| \leq 1\},$$

and  $\Phi \in \text{ev}(B)$  iff there exists an  $x \in B$  such that  $\text{ev}(x) = \Phi$ . Therefore

$$\text{ev}(B)^\circ = \{\phi \in E^\beta \mid \forall x \in B. |\text{ev}(x)(\phi)| \leq 1\} = \{\phi \in E^\beta \mid \forall x \in B. |\phi(x)| \leq 1\}.$$

This is equal to  $\text{Ball}(E^\beta) = B^\circ$  (the polar being with respect to the  $(E, E^\beta)$  duality this time).

Taking polars, we get that  $C = \text{Ball}(E^\beta)^\circ = \text{ev}(B)^{\circ\circ}$ . Since  $\text{ev}$  is continuous,  $\text{ev}(B)$  is a compact, hence closed, subset of  $E^{\beta\sigma}$ , and it is also absolutely

convex by the linearity of  $\text{ev}$ . So  $\text{ev}(B)^{\circ\circ} = \text{ev}(B)$ , and we conclude that  $C = \text{ev}(B)$ . By linearity of  $\text{ev}$  we also obtain  $\alpha C = \text{ev}(\alpha B)$  for all  $\alpha \in \mathbb{R}_{>0}$ .

Now, let  $\Phi \in E^{\beta\sigma}$ . Since  $C$  is absorbent, being the unit ball of a norm, there is an  $\alpha \in \mathbb{R}_{>0}$  such that  $\Phi \in \alpha C$ . Since  $\text{ev}(\alpha B) = \alpha C$ , there is an  $x \in \alpha B \subseteq E$  such that  $\text{ev}(x) = \Phi$ , so we have shown that  $\text{ev}$  is surjective.

Since  $\text{ev}$  is a continuous bijection, to show that it is a homeomorphism we only need to show that it is an open mapping. So let  $U \in \mathcal{T}$ . For all  $\alpha \in \mathbb{R}_{>0}$ ,  $\text{ev}|_{\alpha B}$  is a continuous bijection of compact Hausdorff spaces from  $\alpha B$  to  $\alpha C$ , and therefore an open mapping, so  $\text{ev}(U \cap \alpha B)$  is relatively open in  $\alpha C$ , so is equal to  $V_\alpha \cap \alpha C$  for some  $V_\alpha$  that is open in  $E^{\beta\sigma}$ . Therefore for all  $\alpha \in \mathbb{R}_{>0}$

$$\begin{aligned} \text{ev}(U) \cap \alpha C &= \text{ev}(U) \cap \text{ev}(\alpha B) \\ &= \text{ev}(U \cap \alpha B) && \text{ev bijective} \\ &= V_\alpha \cap \alpha C, \end{aligned}$$

so by Lemma 3.2.10,  $\text{ev}(U)$  is open in  $E^{\beta\sigma}$ .  $\square$

The preceding proposition shows that our redefinition of Smith space agrees with Akbarov's [2, Theorem 4.11]. Results of the above nature go back to Dixmier's fundamental work [25, Théorème 19], where instead of a dealing with a topology on  $E$  one chose a subspace of the dual, and Ng's improvement of this result [90, Theorem 1].

Propositions 3.2.13, 3.2.14 and 3.2.17 can also be derived from Cooper's results about Saks spaces [21, Chapter I, 1.13, 4.1 and 4.2]. This is via a different approach, taking the finest *linear* topology agreeing with the original topology on a given family of compact sets, and deducing that this is also the finest general topology from the Banach-Dieudonné theorem.

**Corollary 3.2.18.** *For each Smith space  $(E, \mathcal{T}, B)$ , the underlying normed space  $(E, \|\cdot\|_B)$  is complete. There are forgetful functors  $U_1 : \mathbf{Smith}_1 \rightarrow \mathbf{Ban}_1$  and  $U_\infty : \mathbf{Smith} \rightarrow \mathbf{Ban}$ .*

*Proof.* By Proposition 3.2.17,  $\text{ev}$  is an isomorphism between  $E$  and  $E^{\beta\sigma}$ , and the unit ball  $B$  of  $E$  is mapped to the unit ball  $C$  of  $E^{\beta\sigma}$ . Therefore  $\text{ev} : (E, \|\cdot\|_B) \rightarrow (E^{\beta\sigma}, \|\cdot\|_C)$  is an isometry of normed spaces. Since  $(E^{\beta\sigma}, \|\cdot\|_C)$  is the dual of a Banach space, it is a Banach space [28, Corollary II.3.9], and therefore  $E$  is.

To show that  $U_1$  exists, observe that a map  $f : (E, \mathcal{T}, B) \rightarrow (F, \mathcal{S}, C)$  in  $\mathbf{Smith}_1$  maps  $B$  into  $C$  and so is bounded of norm  $\leq 1$  by Lemma 0.1.8.

For  $U_\infty$ , we need a different argument. Let  $f : (E, \mathcal{T}, B) \rightarrow (F, \mathcal{S}, C)$  be a continuous map of Smith spaces. The set  $f(B) \subseteq F$  is compact, and therefore

there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $f(B) \subseteq \alpha C$  (Proposition 3.2.14). Therefore  $f$  is bounded with norm  $\leq \alpha$  (Lemma 0.1.8).  $\square$

**Corollary 3.2.19.** *If  $f : (E, \mathcal{T}, B) \rightarrow (F, \mathcal{S}, C)$  is a continuous linear bijection of Smith spaces, it is an isomorphism, i.e. the inverse is continuous.*

*Proof.* By Corollary 3.2.18  $f$  is a bounded surjective map of Banach spaces. Therefore it is an open mapping (i.e. the image of an open set is open) by the open mapping theorem [20, §III.12.1] [110, III.2.1 Corollary 1] [28, Theorem II.2.1]. The open unit ball of  $E$  contains zero, so 0 is in the  $\|\cdot\|_C$ -interior of  $f(B)$ . Therefore there exists a  $\beta \in \mathbb{R}_{>0}$  such that  $\beta C \subseteq f(B)$ .

To show that  $f^{-1}$  is continuous, it suffices to show that  $f^{-1}|_C$  is continuous. First, observe that as  $\beta C \subseteq f(B)$ ,  $C \subseteq \beta^{-1}f(B) = f(\beta^{-1}B)$ . Now, the restricted map  $f|_{\beta^{-1}B} : \beta^{-1}B \rightarrow f(\beta^{-1}B)$  is a continuous bijection of compact Hausdorff spaces, and therefore a homeomorphism. So for any open set  $U \subseteq E$ , there exists an open set  $V \subseteq F$  such that  $f|_{\beta^{-1}B}(U \cap \beta^{-1}B) = V \cap f(\beta^{-1}B)$ . As  $f$  is a bijection,  $f(U \cap \beta^{-1}B) = f(U) \cap f(\beta^{-1}B)$ , so we have

$$f(U) \cap f(\beta^{-1}B) = V \cap f(\beta^{-1}B),$$

and therefore  $f(U) \cap C = V \cap C$ , so  $f(U) \cap C = (f^{-1}|_C)^{-1}(U)$  is relatively open in  $C$ . This shows that  $f^{-1}|_C$  is continuous, so  $f^{-1}$  is continuous by Proposition 3.2.15.  $\square$

### 3.2.1 $\beta$ and $\sigma$ as functors

We now show how to define the strong dual functor  ${}^{-\beta} : \mathbf{Smith} \rightarrow \mathbf{Ban}^{\text{op}}$  and the weak dual functor  ${}^{-\sigma} : \mathbf{Normed}^{\text{op}} \rightarrow \mathbf{Smith}$ , extending their definition on objects.

Let  $f : (E, \mathcal{T}, B) \rightarrow (F, \mathcal{S}, C)$  be a continuous linear map of Smith spaces. Define

$$f^\beta(\psi) = \psi \circ f,$$

where  $\psi \in F^\beta$ .

**Proposition 3.2.20.** *The above defines a functor  ${}^{-\beta} : \mathbf{Smith} \rightarrow \mathbf{Ban}^{\text{op}}$  and  $\mathbf{Smith}_1 \rightarrow \mathbf{Ban}_1^{\text{op}}$ .*

*Proof.* If  $\psi \in F^\beta$ , then as it is the composite of two continuous linear maps,  $\psi \circ f$  is continuous and linear, so is an element of  $E^\beta$ .

If  $\alpha\phi + \beta\psi$  is a linear combination in  $F^\beta$ , then for all  $x \in E$

$$\begin{aligned} f^\beta(\alpha\phi + \beta\psi)(x) &= (\alpha\phi + \beta\psi)(x) = \alpha\phi(x) + \beta\psi(x) = \alpha f^\beta(\phi)(x) + \beta f^\beta(\psi)(x) \\ &= (\alpha f^\beta(\phi) + \beta f^\beta(\psi))(x), \end{aligned}$$

so  $f^\beta$  is a linear map.

Since  $f$  is continuous,  $f(B) \subseteq F$  is compact, so there exists an  $\alpha \in \mathbb{R}_{>0}$  such that  $f(B) \subseteq \alpha C$  by Proposition 3.2.14 (in the case that  $f \in \mathbf{Smith}_1(E, F)$  we already know  $f(B) \subseteq C$  so do not need to prove this).

We show that  $f^\beta(C^\circ) \subseteq \alpha B^\circ$ . If  $\psi \in C^\circ$  we have that for all  $x \in C$ ,  $\psi(x) \leq 1$ . Since  $f(B) \subseteq \alpha C$ , we have  $f(\alpha^{-1}B) \subseteq C$ , by linearity. Taking these two facts together, we have that

$$\begin{aligned} \forall x \in \alpha^{-1}B. \psi(f(x)) &\leq 1 \Rightarrow \forall x \in \alpha^{-1}B. f^\beta(\psi)(x) \leq 1 \\ &\Leftrightarrow f^\beta(\psi) \in \alpha^{-1}B^\circ = \alpha B^\circ, \end{aligned}$$

by Lemma 0.3.11 (ii). We then use Lemma 0.1.8 to deduce that  $\|f^\beta\| \leq \alpha$ , so  $f^\beta$  is bounded, and therefore a morphism in  $\mathbf{Ban}$  from  $F^\beta \rightarrow E^\beta$ . If  $f \in \mathbf{Smith}_1(E, F)$ , then previous argument shows  $f^\sigma(C^\circ) \subseteq B^\circ$  so  $\|f^\beta\| \leq 1$  and  $f^\beta \in \mathbf{Ban}_1(F^\beta, E^\beta)$ .

Let  $\text{id}_E$  be an identity map of Smith spaces. Then if  $\phi \in E^\beta$ , we have  $\text{id}_E^\beta(\phi) = \phi \circ \text{id}_E = \phi$ , so  $\text{id}_E^\beta = \text{id}_{E^\beta}$ . If  $f : E \rightarrow F$  and  $g : F \rightarrow G$  are maps of Smith spaces and  $\psi \in G^\beta$

$$(g \circ f)^\beta(\psi) = \psi \circ g \circ f = f^\beta(\psi \circ g) = (f^\beta \circ g^\beta)(\psi),$$

which finishes the proof that  ${}^\beta$  is a contravariant functor.  $\square$

Now let  $f : E \rightarrow F$  be a bounded (or, equivalently, continuous) map of normed spaces. Define

$$f^\sigma(\psi) = \psi \circ f,$$

where  $\psi \in F^\sigma$ .

**Proposition 3.2.21.** *The above defines a functor  ${}^\sigma : \mathbf{Normed}^{\text{op}} \rightarrow \mathbf{Smith}$  and  $\mathbf{Normed}_1^{\text{op}} \rightarrow \mathbf{Smith}_1$ .*

*Proof.* Since linearity and continuity of functions are preserved under composition, we have that  $\psi \circ f$  is always an element of  $E^\sigma$  for any  $f \in \mathbf{Normed}(E, F)$  and  $\psi \in F^\sigma$ . The proof that  $f^\sigma$  is linear is identical to the proof of the linearity of  $f^\beta$  in Proposition 3.2.20, so is omitted.

To show  $f^\sigma$  is continuous from the topology  $\sigma(F^\sigma, F)_b$  to  $\sigma(E^\sigma, E)_b$ , we first show that it is continuous from  $\sigma(F^\sigma, F)$  to  $\sigma(E^\sigma, E)$ . We use the neighbourhood definition of continuity. Let  $N_x$ , where  $x \in E$ , be a subbasic neighbourhood in  $F$  for the  $\sigma(E^\sigma, E)$ . We can show that  $N_{f(x)} \subseteq (f^\sigma)^{-1}(N_x)$  as follows:

$$\begin{aligned} \psi \in N_{f(x)} &\Leftrightarrow |\psi(f(x))| < 1 \Leftrightarrow |(\psi \circ f)(x)| < 1 \Leftrightarrow f^\sigma(\psi) \in N_x \\ &\Leftrightarrow \psi \in (f^\sigma)^{-1}(N_x). \end{aligned}$$

Since preimages preserve intersections, we have that the preimage of every basic 0-neighbourhood in the  $\sigma(E^\sigma, E)$ -topology is a 0-neighbourhood in the  $\sigma(F^\sigma, F)$ -topology, establishing continuity with respect to these topologies.

To show continuity for the corresponding bounded weak-\* topologies, we can first see that, as  $\sigma(F^\sigma, F)$  is coarser than  $\sigma(F^\sigma, F)_b$ ,  $f^\sigma$  is continuous from  $(F, \sigma(F^\sigma, F)_b)$  to  $(E, \sigma(E^\sigma, E))$ . We therefore know that  $f^\sigma|_{C^\circ}$  is continuous with the same topologies. Now, since  $f$  is bounded, we can apply the same argument used in Proposition 3.2.20 to deduce  $f^\beta(C^\circ) \subseteq \alpha B^\circ$  from  $f(B) \subseteq \alpha C$  to  $f^\sigma$  instead and deduce that  $f^\sigma(C^\circ) \subseteq \alpha B^\circ$ , with  $\alpha \leq 1$  in the case that  $f \in \mathbf{Normed}_1(E, F)$ . We then apply Corollary 3.2.16 to deduce that  $f^\sigma$  is continuous from  $(F, \sigma(F^\sigma, F)_b)$  to  $(E, \sigma(E^\sigma, E)_b)$ . If  $f \in \mathbf{Normed}_1(E, F)$  this also shows  $f^\sigma \in \mathbf{Smith}_1(F^\sigma, E^\sigma)$ .

The proof of preservation of identity maps and composition of maps is similar to that in 3.2.20 and so is omitted.  $\square$

We now define  $\eta_E : E \rightarrow E^{\beta\sigma}$  in **Smith** and  $\epsilon_E : E \rightarrow E^{\sigma\beta}$  in **Normed** as

$$\begin{aligned} \eta_E(x)(\phi) &= \phi(x) && \text{for } x \in E \text{ and } \phi \in E^\beta \\ \epsilon_E(x)(\phi) &= \phi(x) && \text{for } x \in E \text{ and } \phi \in E^\sigma \end{aligned}$$

**Theorem 3.2.22.** *The families of maps  $\eta_E$  and  $\epsilon_E$  define the unit and counit of an adjunction  $\beta \dashv \sigma$ , for **Normed**<sup>op</sup> and **Smith** and also **Normed**<sub>1</sub><sup>op</sup> and **Smith**<sub>1</sub>. The map  $\eta_E$  is an isomorphism, while  $\epsilon_E$  is an isomorphism iff  $E$  is complete.*

*Proof.* First, observe that the definition of  $\eta_E$  coincides with that of  $\text{ev}$  in Proposition 3.2.17 and that  $\text{ev}$  is proven there to be defined with the correct codomain and to be an isomorphism in **Smith**. In the course of the proof it is shown that for  $(E, \mathcal{T}, B)$  a Smith space,  $\text{ev}(B)$  is the unit ball of  $E^{\beta\sigma}$ , there called  $C$ , so it is also an isomorphism in **Smith**<sub>1</sub>.

We therefore move on to proving that  $\eta_E$  is natural. Let  $f \in \mathbf{Smith}(E, F)$ . We want to show that

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \eta_E \downarrow & & \downarrow \eta_F \\ E^{\beta\sigma} & \xrightarrow{f^{\beta\sigma}} & F^{\beta\sigma} \end{array}$$

commutes, which is to say, that if  $x \in E$  and  $\psi \in F^\beta$ , then

$$\eta_F(f(x))(\psi) = f^{\beta\sigma}(\eta_E(x))(\psi)$$

For the left hand side, we have  $\eta_F(f(x))(\psi) = \psi(f(x))$ . For the right hand side, we have

$$f^{\beta\sigma}(\eta_E(x))(\psi) = \eta_E(x)(f^\beta(\psi)) = f^\beta(\psi)(x) = \psi(f(x)).$$

Therefore the diagram commutes, and we have shown  $\eta$  is natural.

We now move on to showing that  $\epsilon_E$  is defined correctly and is natural. The map  $\epsilon_E$  has the same definition as  $\langle x, - \rangle$  for the pairing between the space  $E$  and its continuous dual  $E^*$  (Proposition 0.3.1). Therefore  $\epsilon_E(x)$  is linear and continuous for the  $\sigma(E^\sigma, E)$ -topology on  $E^\sigma$ . Since  $\sigma(E^\sigma, E)_b$  is finer than  $\sigma(E^\sigma, E)$ ,  $\epsilon_E(x)$  is also continuous in that topology, and therefore is an element of  $E^{\sigma\beta}$ . We also have that  $\epsilon_E$  is linear by using Proposition 0.3.1 again.

We show that  $\epsilon_E$  is bounded as follows. Let  $B$  be the unit ball of  $E$ ,  $C = B^\circ$  the unit ball of  $E^\sigma$ , and  $D = C^\circ$  the unit ball of  $E^{\sigma\beta}$ . We want to show that  $\epsilon_E(B) \subseteq D$ . So let  $x \in B$ . By Corollary 0.3.10,  $B = B^{\circ\circ} = C^\circ$ . Then

$$\forall \phi \in C. |\phi(x)| \leq 1 \Leftrightarrow \forall \phi \in C. |\epsilon_E(x)(\phi)| \leq 1 \Leftrightarrow \epsilon_E(x) \in C^\circ = D.$$

So  $\epsilon_E$  is bounded with norm  $\leq 1$ , therefore a map in  $\mathbf{Normed}_1$ . If  $\epsilon_E$  is bijective, the above argument also shows its inverse has norm  $\leq 1$ , so it would be an isomorphism in  $\mathbf{Normed}_1$ .

We show that  $\epsilon_E$  is bijective, and therefore an isomorphism in  $\mathbf{Normed}_1$ , iff  $E$  is a Banach space. In Proposition 3.2.13 we have seen that  $E^{\sigma\beta}$  is the completion of  $(E^\sigma, \sigma(E^\sigma, E), B^\circ)^\beta$ , under the inclusion mapping. By Proposition 0.3.2,  $\epsilon_E$  maps  $E$  bijectively onto  $(E^\sigma, \sigma(E^\sigma, E))^*$ , and we showed in the previous paragraph that this mapping preserves the norm. Therefore  $\epsilon_E$  shows that  $E^{\sigma\beta}$  is a completion of  $E$ . This is an isomorphism iff  $E$  is already complete, *i.e.* a Banach space.

The proof that  $\epsilon$  is natural is similar to the proof that  $\eta$  is natural, so is omitted.

We now show that the following diagrams commute, which are the unit-counit diagrams for showing that  $\beta \dashv \sigma$  (Theorem 0.4.1 (v)).

$$\begin{array}{ccc}
 E^\beta & \xrightarrow{\epsilon_{E^\beta}} & E^{\beta\sigma\beta} \\
 \searrow \text{id}_{E^\beta} & & \downarrow \eta_E^\beta \\
 & & E^\beta
 \end{array}
 \qquad
 \begin{array}{ccc}
 E^\sigma & \xrightarrow{\eta_{E^\sigma}} & E^{\sigma\beta\sigma} \\
 \searrow \text{id}_{E^\sigma} & & \downarrow \epsilon_E^\sigma \\
 & & E^\sigma
 \end{array}$$

Note that  $E$  is a Smith space in the left triangle, while the triangle itself is in **Normed**, and so is reversed from its usual appearance. In the triangle on the right,  $E$  is a normed space and the triangle is in **Smith**.

To show the left triangle commutes, let  $\phi \in E^\beta$  and  $x \in E$ . Then

$$\eta_E^\beta(\epsilon_{E^\beta}(\phi))(x) = \epsilon_{E^\beta}(\phi)(\eta_E(x)) = \eta_E(x)(\phi) = \phi(x).$$

As this holds for all  $x \in X$  and  $\phi \in E^\beta$ , we get  $\eta_E^\beta \circ \epsilon_{E^\beta}(\phi) = \text{id}_{E^\beta}$  as required.

The proof that the right triangle commutes is similar, with  $\sigma$  replacing  $\beta$  and the rôles of  $\eta$  and  $\epsilon$  reversed, so is omitted.

During the proof that  $\eta$  and  $\epsilon$  are well defined, we already showed that  $\eta$  is always an isomorphism and  $\epsilon_E$  an isomorphism whenever  $E$  is Banach.  $\square$

The following corollary is immediate.

**Corollary 3.2.23.** *The functors  $-^\beta$  and  $-\sigma$  define equivalences*

$$\begin{aligned}
 \mathbf{Ban}^{\text{op}} &\simeq \mathbf{Smith} \\
 \mathbf{Ban}_1^{\text{op}} &\simeq \mathbf{Smith}_1.
 \end{aligned}$$

We also have

**Corollary 3.2.24.** *If  $(E, \mathcal{T}, B)$  and  $(F, \mathcal{S}, C)$  are Smith spaces, then a linear map  $f : E \rightarrow F$  is continuous from  $\mathcal{T}$  to  $\mathcal{S}$  iff it is continuous from  $\sigma(E, E^\beta)$  to  $\sigma(F, F^\beta)$ . A special case is that if  $E_*, F_*$  are Banach spaces, then a linear map  $E_*^\sigma \rightarrow F_*^\sigma$  is continuous on the Smith space topologies iff it is weak-\* continuous.*

*Proof.* Suppose  $f : E \rightarrow F$  is continuous from  $\sigma(E, E^\beta) \rightarrow \sigma(F, F^\beta)$ . By Lemma 3.2.7,  $B$  is  $\sigma(E, E^\beta)$ -compact, so  $f(B)$  is  $\sigma(F, F^\beta)$ -compact, and therefore  $\mathcal{S}$ -compact because  $\mathcal{S}$  is a finer topology. By Proposition 3.2.14,  $f(B) \subseteq$



$\alpha C$  for some  $\alpha \in \mathbb{R}_{>0}$ , and so by Corollary 3.2.16,  $f$  is continuous from  $\mathcal{T}$  to  $\mathcal{S}$ .

If, on the other hand, we start with  $f : E \rightarrow F$  being continuous from  $\mathcal{T}$  to  $\mathcal{S}$ , we have that  $f^\beta : F^\beta \rightarrow E^\beta$ . If we consider the usual pairings between the spaces  $E, F$  and their duals  $E^\beta, F^\beta$ , we have, for all  $\phi \in F^\beta$  and  $x \in E$

$$\langle f^\beta(\phi), x \rangle = \langle \phi \circ f, x \rangle = \phi(f(x)) = \langle \phi, f(x) \rangle,$$

so by Proposition 0.3.3  $f$  is continuous from  $\sigma(E, E^\beta)$  to  $\sigma(F, F^\beta)$ .

The statement for  $E_*$  and  $F_*$  follows from the fact that the counit map  $\epsilon_{E_*} : E_* \rightarrow E_*^{\sigma\beta}$  is an isometry of Banach spaces (Theorem 3.2.22).  $\square$

We prove one more fact that we will need later.

**Proposition 3.2.25.** *Let  $f : E \rightarrow F$  be a bounded map of Banach spaces.*

(i) *If  $f(E)$  is dense in  $F$ , then  $f^\sigma$  is injective.*

(ii) *If  $f$  is injective,  $f^\sigma(F^\sigma)$  is dense in  $E^\sigma$ .*

*Proof.*

(i) Let  $\phi, \psi \in F^\sigma$  such that  $f^\sigma(\phi) = f^\sigma(\psi)$ . This means  $\phi$  and  $\psi$  agree on all elements of  $f(E)$ , a dense subset of  $F$ . As they are continuous,  $\phi = \psi$ .

(ii) The set  $f^\sigma(F^\sigma)$  is a subspace of  $E^\sigma$ , so is a convex set. Therefore its closure in the Smith topology of  $E^\sigma$  equals its closure in  $E^\sigma$ 's weak topology  $\sigma(E^\sigma, E^{\sigma\beta}) = \sigma(E^\sigma, E)$  (Proposition 0.3.4), which equals its bipolar by Corollary 0.3.10. So

$$\begin{aligned} \text{cl}(f^\sigma(F^\sigma)) &= f^\sigma(F^\sigma)^{\circ\circ} \\ &= f^{-1}(F^{\sigma\circ})^\circ && \text{Lemma 0.3.14} \\ &= f^{-1}(\{0\})^\circ \\ &= \{0\}^\circ && f \text{ injective} \\ &= F^\sigma. \end{aligned}$$

$\square$

### 3.3 Compact Convex Sets and Smith Base-Norm Spaces

We first define Smith base-norm spaces. A *Smith base-norm space* is a quadruple  $(E, \mathcal{T}, E_+, \tau)$ , where  $(E, \mathcal{T})$  is a locally convex topology,  $E_+$  is a closed positive cone in this topology, and  $\tau$  is a  $\mathcal{T}$ -continuous map  $E \rightarrow \mathbb{R}$  such that  $(E, E_+, \tau)$  is a base-norm space, and  $(E, \mathcal{T}, \text{absco}(B_E))$  is a Smith space. A trace-preserving morphism  $f : (E, \mathcal{T}, E_+, \tau) \rightarrow (F, \mathcal{S}, F_+, \sigma)$  of Smith base-norm spaces is a continuous linear map that is a trace-preserving morphism of the underlying base-norm spaces. Trace-reducing maps are defined in a similar manner and Smith spaces with each kind of map form the categories **SBNS** and **SBNS**<sub>≤1</sub>, respectively.

In the following, we will often need to consider, given a topological vector space  $E$ , the map  $c : \mathbb{R} \times E \times E \rightarrow E$  defined by

$$c(\alpha, x, y) = \alpha x + (1 - \alpha)y. \quad (3.5)$$

This mapping can be written as

$$c = + \circ ((-\cdot -) \times (-\cdot -)) \circ (\text{id}_R \times \sigma_{\mathbb{R}, E} \times \text{id}_E) \circ ((\text{id}_{\mathbb{R}}, 1 - \cdot) \times \text{id}_{E \times E}),$$

which is therefore a continuous map by the definition of a topological vector space.

**Lemma 3.3.1.** *Let  $(E, \mathcal{T})$  be a topological vector space, and  $X \subseteq E$  a compact convex subset. Let  $B = \text{absco}(X)$  and  $f : (E, \mathcal{T}) \rightarrow (F, \mathcal{S})$  be a linear map such that  $f|_X$  is continuous. Then  $f|_B$  is continuous.*

*Proof.* We first define  $g : \mathbb{R} \times E \times E$  such that

$$\begin{array}{ccc} \mathbb{R} \times E \times E & & \\ \downarrow c & \searrow g & \\ E & \xrightarrow{f} & F \end{array}$$

commutes.

Define  $g(\alpha, x, y) = \alpha f(x) + (1 - \alpha)f(y)$ . We see that  $f(c(\alpha, x, y)) = f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y) = g(\alpha, x, y)$  by linearity of  $f$ . We use Corollary A.2.2 to prove that  $g|_{[0,1] \times X \times X}$  is continuous. Let  $(\alpha_i, x_i, y_i)_{i \in I}$  be a net converging in the product topology to  $(\alpha, x, y)$ , everything being contained in

$[0, 1] \times X \times X$ . Then

$$\begin{aligned}
& \lim_{i \in I} g(\alpha_i, x_i, y_i) \\
&= \lim_{i \in I} \alpha_i f(x_i) + (1 - \alpha_i) f(y_i) \\
&= \lim_{i \in I} \alpha_i f(x_i) + \lim_{i \in I} (1 - \alpha_i) f(y_i) && + \text{continuous} \\
&= \left( \lim_{i \in I} \alpha_i \right) \cdot \left( \lim_{i \in I} f(x_i) \right) + \left( \lim_{i \in I} (1 - \alpha_i) \right) \cdot \left( \lim_{i \in I} f(y_i) \right) && \cdot \text{continuous} \\
&= \alpha f(x) + \left( \lim_{i \in I} (1 - \alpha_i) \right) f(y) && f|_X \text{ continuous} \\
&= \alpha f(x) + (1 - \alpha) f(y) && +, - \text{continuous} \\
&= g(\alpha, x, y),
\end{aligned}$$

which establishes the continuity.

To show that  $f|_B$  is continuous, we first show that for each closed set  $C \subseteq F$ ,  $f^{-1}(C) \cap B$  is closed. We know that  $g^{-1}(C) \cap [0, 1] \times X \times X$  is closed, by the continuity of  $g|_{[0,1] \times X \times X}$ . As  $g = f \circ c$ , this implies that the set  $c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X$  is closed, and as it is a closed subset of a compact space, it is compact. Therefore  $c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$  is compact, and therefore closed. If we show  $f^{-1}(C) \cap B = c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$ , we will have shown it is closed. We do this by showing an inclusion in each direction.

If  $x' \in f^{-1}(C) \cap B$ , then as  $B = \text{co}(-X \cup X)$ , we have that there are  $x, y \in X$  and  $\alpha \in [0, 1]$  such that  $\alpha x + (1 - \alpha)y = x'$ , *i.e.*  $c(\alpha, x, y) = x'$ . Therefore  $(\alpha, x, y) \in c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X$ , so  $x' \in c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$ .

For the other direction, if  $x' \in c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$ , there exist  $(\alpha, x, y) \in c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X$  such that  $x' = c(\alpha, x, y)$ , so  $c(\alpha, x, y) \in f^{-1}(C)$  and  $c(\alpha, x, y) = \alpha x + (1 - \alpha)y \in \text{co}(-X \cup X) = B$ , so  $x' \in f^{-1}(C) \cap B$ .

Now, let  $V \subseteq F$  be an open set, so  $F \setminus V$  is closed. Then  $f^{-1}(F \setminus V) \cap B$  is closed in  $E$ , and so  $E \setminus (f^{-1}(F \setminus V) \cap B)$  is open in  $E$ . Now

$$\begin{aligned}
(E \setminus (f^{-1}(F \setminus V) \cap B)) \cap B &= B \setminus (f^{-1}(F \setminus V) \cap B) \\
&= B \setminus ((E \setminus f^{-1}(V)) \cap B) \\
&= B \setminus (B \setminus f^{-1}(V)) \\
&= f^{-1}(V) \cap B,
\end{aligned}$$

so  $f^{-1}(V) \cap B$  is relatively open in  $B$ , as required.  $\square$

The following proposition is a version of [33, Theorem 4] adapted to compact convex sets and Smith spaces instead of locally compact cones.

**Proposition 3.3.2.** *If  $(E, E_+, \tau)$  is a pre-base-norm space,  $\mathcal{T}$  a locally convex topology on  $E$  in which  $B_E$  is compact, then  $(E, \mathcal{T}_b, E_+, \tau)$  is a Smith base-norm space, where  $\mathcal{T}_b$  is taken with respect to the compact barrel  $\text{absco}(B_E)$ .*

*Proof.* If  $B_E$  is empty, then by Lemma 2.2.1  $E = 0$  and the result holds tautologically as there is only one topology on  $E$ , which is Smith. Therefore we now assume that  $B_E \neq \emptyset$ . As it is a product of compact sets, the set  $[0, 1] \times B_E \times B_E \subseteq \mathbb{R} \times E \times E$  is compact, where each  $E$  has  $\mathcal{T}$  as its topology. Because  $B_E$  is already convex,  $c([0, 1] \times B_E \times B_E) = \text{co}(-B_E \cup B_E)$  (see (3.5)), and this is  $\text{absco}(B_E)$  (Lemma 0.1.1). As it is the image of a compact set under a continuous map, we have shown  $\text{absco}(B_E)$  is compact in  $\mathcal{T}$ , and therefore radially compact, so  $(E, E_+, \tau)$  is a base-norm space. The set  $\text{absco}(B_E)$  is also absolutely convex, and is absorbent by Lemma 2.2.3, so is a compact barrel. We can therefore define  $\mathcal{T}_b$  with respect to it, and obtain a Smith space  $(E, \mathcal{T}_b, \text{absco}(B_E))$  (Proposition 3.2.9).

We can show that  $E_+$  is closed as follows. Let  $\alpha \in \mathbb{R}_{>0}$ . Then, by Corollary 2.2.9,  $E_+ \cap \alpha \text{absco}(B_E) = \alpha \text{co}(\{0\} \cup B_E)$ . Now  $\alpha \text{co}(\{0\} \cup B_E)$  is the image of  $[0, \alpha] \times B_E$  under the continuous map  $- \cdot - : \mathbb{R} \times E \rightarrow E$ , so is compact, and therefore closed. So  $E_+ \cap \alpha \text{absco}(B_E)$  is relatively closed in  $\text{absco}(B_E)$  for all  $\alpha \in \mathbb{R}_{>0}$ , which by Lemma 3.2.10 implies it is closed. The map  $\tau$  is the constant 1 function when restricted to  $B_E$ , so  $\tau_{B_E}$  is continuous. By 3.3.1,  $\tau|_{\text{absco}(B_E)}$  is continuous. Therefore  $\tau$  is continuous (in  $\mathcal{T}_b$ ) by Proposition 3.2.15, and so  $(E, \mathcal{T}_b, E_+, \tau)$  is a Smith base-norm space.  $\square$

Recall the category **CCL**, which has pairs  $(E, X)$  as objects, where  $E$  is a locally convex space and  $X \subseteq E$  a compact convex set, and where maps  $(E, X) \rightarrow (F, Y)$  are simply affine continuous maps  $X \rightarrow Y$ . We have a functor  $B : \mathbf{SBNS} \rightarrow \mathbf{CCL}$  defined on objects as  $B(E) = (E, B_E)$  and on maps as restriction, similar to the definition in and after Proposition 2.2.13 for pre-base-norm spaces and **BConv**.

**Proposition 3.3.3.** *The functor  $B : \mathbf{SBNS} \rightarrow \mathbf{CCL}$  is an equivalence of categories.*

*Proof.* We show that  $B$  is faithful, full and essentially surjective. The proof that  $B$  is faithful is the same as the proof that  $B^D : \mathbf{PreBNS} \rightarrow \mathcal{EM}(D)$  is faithful in Proposition 2.4.8.

To show it is full, let  $(E, \mathcal{T}, E_+, \tau)$  and  $(F, \mathcal{S}, F_+, \sigma)$  be Smith base-norm spaces, and let  $g : B_E \rightarrow B_F$  be a continuous affine map. It is therefore

an affine map and so, by Proposition 2.4.8, it extends to a trace-preserving (linear) map  $f : (E, E_+, \tau) \rightarrow (F, F_+, \sigma)$ . Since  $f|_{B_E} = g$ , we know that  $f|_{B_E}$  is continuous from  $B_E \rightarrow (F, \mathcal{S})$ , so by Lemma 3.3.1  $f|_{\text{absco}(B_E)}$  is continuous, and since  $(E, \mathcal{T}, \text{absco}(B_E))$  is a Smith space we apply Proposition 3.2.15 to conclude that  $f$  is continuous, and therefore a map in **SBNS**.

To show that it is essentially surjective, let  $(E, X)$  be an object of **CCL**,  $\mathcal{T}$  being the topology on  $E$ , which as all compact sets are bounded (Lemma 0.1.14) is also an element of **BConv**. By Proposition 2.2.13, there exists a pre-base-norm space  $(F, F_+, \tau)$  with locally convex topology  $\mathcal{S}$  and a **BConv** isomorphism  $i : (E, X) \rightarrow (F, B_F)$  that is relatively continuous from  $\mathcal{T}|_X$  to  $\mathcal{S}|_{B_F}$ . By Proposition 3.3.2,  $(F, \mathcal{S}_b, F_+, \tau)$  is a Smith space, whose topology agrees with  $\mathcal{S}$  on  $\text{absco}(B_F)$  and therefore on  $B_F$  itself. This means the identity mapping  $(F, \mathcal{S}, B_F) \rightarrow (F, \mathcal{S}_b, B_F)$  is an isomorphism in **CCL**, so composing it with  $i : (E, X) \rightarrow (F, B_F)$  proves that  $B$  is essentially surjective.  $\square$

By Theorem 0.4.3, we can find a functor  $\text{Emb} : \mathbf{CCL} \rightarrow \mathbf{SBNS}$  (for embedding) such that  $B$  and  $\text{Emb}$  are part of an adjoint equivalence.

### 3.3.1 Continuous Affine Functions and the Strong Dual

The following is a standard construction (see e.g. [4, §I.1]). If  $(E, X) \in \mathbf{CCL}$ , we define

$$\text{CAff}(X) = \{a : X \rightarrow \mathbb{R} \mid a \text{ affine and continuous}\}.$$

We take  $\text{CAff}(X)_+$  to be elements of  $\text{CAff}(X)$  with range inside  $\mathbb{R}_{\geq 0}$ , and its unit to be the constant function with value 1.

Given  $f : (E, X) \rightarrow (F, Y)$  in **CCL**, we can define

$$\text{CAff}(f)(b) = b \circ f.$$

**Proposition 3.3.4.** *CAff is a functor from **CCL** to **BOUS**<sup>op</sup>.*

*Proof.* We first show that  $\text{CAff}(X)$  is a Banach order-unit space for any  $(E, X) \in \mathbf{CCL}$ . We have that  $\text{CAff}(X) \subseteq C(X)$ , where  $C(X)$  is taken as the real-valued continuous functions. That linear combinations of affine functions are affine was already proven in the first part of Proposition 2.4.15, and linear combinations of continuous functions are continuous because addition and multiplication of real numbers are continuous, so  $\text{CAff}(X)$  is a linear subspace of  $C(X)$ . This proves that  $\text{CAff}(X)$  is a vector space under the pointwise operations.

We have that  $\text{CAff}(X)_+$  is a cone because  $C(X, \mathbb{R}_{\geq 0})$  is a cone, the unit element is affine by an argument in Proposition 2.4.15 and continuous because it is constant. The unit element is a strong archimedean unit because it is a strong archimedean unit in  $C(X)$ , so  $\text{CAff}(X)$  is an order-unit space.

To see that  $\text{CAff}(X)$  is a Banach space, we show that it is closed in  $C(X)$ , which is a Banach space. So if  $(a_i)_{i \in \mathbb{N}}$  is a sequence of elements of  $\text{CAff}(X)$  converging in norm, then we know from the proof in Proposition 2.4.15 that the limit of that sequence is affine. Since  $C(X)$  is a Banach space, the limit is also continuous, so the limit of  $(a_i)$  is an element of  $\text{CAff}(X)$ .

We now show that  $\text{CAff}(f) : \text{CAff}(Y) \rightarrow \text{CAff}(X)$  is well-defined and a positive unital map for all  $f : (E, X) \rightarrow (F, Y)$ . If we take  $b \in \text{CAff}(Y)$ ,  $\text{CAff}(f) = b \circ f$  is an affine map as it is the composite of two affine maps and is continuous because it is the composite of two continuous maps, so is an element of  $\text{CAff}(A)$ . The linearity of  $\text{CAff}(f)$  follows from the pointwiseness of the operations, and the positivity and unitality have the same proof as for Proposition 2.4.16.

Then  $\text{CAff}$  preserves identity maps because  $a \circ \text{id}_X = a$  and preserves composition because  $a \circ (g \circ f) = (a \circ g) \circ f$ .  $\square$

Given a Smith base-norm space  $(E, \mathcal{T}, E_+, \tau)$ , as  $(E, \mathcal{T}, \text{absc}(B_E))$  is a Smith space, the continuous dual is a Banach space  $E^\beta$  (Proposition 3.2.13). We can define  $E_+^\beta$  to be the dual cone of  $E_+$  (a cone rather than a wedge by Lemma 0.3.8) and the unit  $u = \tau$ , which is an element of  $E^\beta$  by the definition of Smith base-norm space. Given  $\phi \in E^\beta$ , we can define  $\rho_E(\phi) \in \text{CAff}(B_E)$  to be  $\phi|_{B_E}$ .

**Proposition 3.3.5.** *The map  $\rho_E : E^\beta \rightarrow \text{CAff}(B_E)$  is a linear isomorphism preserving the positive cone and unit both ways. Therefore  $E^\beta$  is a Banach order-unit space for any Smith base-norm space  $E$ , with the closed unit ball of  $E^\beta$  being  $[-\tau, \tau]$ .*

*Proof.* If  $a \in E^\beta$ , then  $\rho_E(a) \in \text{CAff}(B_E)$ , because if  $a$  is affine and continuous on  $E$ , it is affine and continuous on the subset  $B_E$ . The proof in Proposition 2.4.17 shows that  $\rho|_E$  is linear and injective, without modification. We also know by the proof in Proposition 2.4.17 that every  $a \in \text{BAff}(B_E)$ , and therefore every  $a \in \text{CAff}(B_E)$  (continuous implies bounded as  $B_E$  is compact), extends to a bounded linear map  $a' : E \rightarrow \mathbb{R}$ . Since  $a'$  extends  $a$ ,  $a'|_{B_E}$  is continuous in the Smith topology of  $E$ . By Lemma 3.3.1,  $a'|_{\text{absc}(B_E)}$  is continuous, and we can then apply Proposition 3.2.15 to deduce that  $a'$  is continuous, and therefore an element of  $\text{CAff}(B_E)$ . Therefore  $\rho_E$  is a linear isomorphism.

The proof that it is a positive unital map is the same as in Proposition 2.4.17, as is the proof that the usual dual unit ball is the same as  $[-\tau, \tau]$ . This shows it is an order-unit space, and by Proposition 3.2.13 it is a Banach order-unit space.  $\square$

**Theorem 3.3.6.** *Restricted to continuous trace-preserving maps,  ${}^\beta$  is a functor  $\mathbf{SBNS} \rightarrow \mathbf{BOUS}^{\text{op}}$ . We call this functor  $F^\beta$ . Then  $\rho$  is a natural isomorphism  $F^\beta \Rightarrow \text{CAff} \circ B$ .*

*Proof.* Let  $f : (D, \mathcal{T}, D_+, \tau) \rightarrow (E, \mathcal{S}, E_+, \sigma)$  be a continuous trace-preserving map. The following diagram commutes

$$\begin{array}{ccc} E^\beta & \xrightarrow{\rho_E} & \text{CAff}(B(E)) \\ f^\beta \downarrow & & \downarrow \text{CAff}(B(f)) \\ D^\beta & \xrightarrow{\rho_D} & \text{CAff}(B(D)) \end{array}$$

by essentially the same argument given for the diagram commuting in Theorem 2.4.18. In fact, the proof that  $F^\beta$  is therefore a functor and  $\rho$  a natural transformation is also essentially the same, and therefore is omitted.  $\square$

We can define  $G^\sigma : \mathbf{BOUS}^{\text{op}} \rightarrow \mathbf{SBNS}$  to be  $G$  with the bounded weak-\* topology. That is to say, if  $(A, A_+, u)$  is a Banach order-unit space, then we know that  $G(A, A_+, u)$ , which is  $(A^*, A_+^*, \text{ev}(u))$ , is a base-norm space. Additionally, the unit ball is compact in  $\sigma(A^*, A)$  by Banach-Alaoglu, and since  $B = A_+^* \cap \text{ev}(u)^{-1}(1)$  is a  $\sigma(A^*, A)$ -closed subset of the unit ball,  $B$  is compact. Therefore  $G^\sigma(A, A_+, u) = (A^*, \sigma(A^*, A)_b, A_+^*, \text{ev}(u))$  is a Smith base-norm space by Proposition 3.3.2. On maps  $G^\sigma$  is defined in the same manner as  $G$ , which is to say that if  $f : (A, A_+, u) \rightarrow (B, B_+, v)$  is a positive unital map and  $\psi \in G^\sigma(B)$

$$G^\sigma(f)(\psi) = \psi \circ f$$

**Theorem 3.3.7.**  *$G^\sigma$  is a functor from  $\mathbf{BOUS}^{\text{op}} \rightarrow \mathbf{SBNS}$ . The restriction of the adjoint equivalence defined by  $\sigma$  and  $\beta$  defines an adjoint equivalence  $\mathbf{BOUS}^{\text{op}} \simeq \mathbf{SBNS}$ .*

*Proof.* Let  $f : (A, A_+, u) \rightarrow (B, B_+, v)$  be a positive unital map. Since the definition of  $G^\sigma$  agrees with  $G$ , except in topology, we have that  $G^\sigma(f)$  is a trace-preserving map of base-norm spaces. By Proposition 1.2.8,  $f$  is a map in  $\mathbf{Ban}_1$ , and as the definition of  $G^\sigma$  agrees with  ${}^\sigma$ , we have that  $G^\sigma(f)$  is

a continuous map of Smith spaces by Proposition 3.2.21. The proof that  $G^\sigma$  preserves identity maps and composition then follows from the proof that  $G$  and  $-\sigma$  are functors.

Following Theorem 3.2.22 and Proposition 2.5.3, we define

$$\begin{aligned}\eta_E : E &\rightarrow G^\sigma(F^\beta(E)) & \epsilon_A : A &\rightarrow F^\beta(G^\sigma(A)) \\ \eta_E(x)(a) &= a(x) & \epsilon_A(a)(\phi) &= \phi(a).\end{aligned}$$

The underlying space of  $G^\sigma(F^\beta(E))$  is  $E^{\beta\sigma}$  and the underlying space of  $F^\beta(G^\sigma(A))$  is  $A^{\sigma\beta}$ . By Theorem 3.2.22 these maps are linear homeomorphisms of the underlying topological vector spaces.

We will show that  $\eta_E$  is an isomorphism of Smith base-norm spaces and  $\epsilon_A$  an isomorphism of Banach order-unit spaces.

The proof that  $\eta_E$  is positive is similar to the proof in Proposition 2.5.3, in short, if  $x \in E_+$  and  $a \in E_+^\beta$ , we have  $\eta_E(x)(a) = a(x) \geq 0$ , so  $\eta_E(x) \in E_+^{\beta\sigma}$ . If, on the other hand, we start with  $\Phi \in E_+^{\beta\sigma}$ , we know from the bijectivity of  $\eta_E$  that there exists an  $x \in E$  such that  $\eta_E(x) = \Phi$ . The positivity of  $\Phi$  implies that for all  $a \in E_+^\beta$ ,  $\eta_E(x)(a) \geq 0$ . Expanding the definition, we have that  $a(x) \geq 0$  for all  $a \in E_+^\beta$ . As  $E_+$  is a closed cone and  $E^\beta = E^*$ , we can use Lemma 0.3.15 to deduce that  $x \in E_+$ . We have shown that  $\eta_E(E_+) = E_+^{\beta\sigma}$ , and therefore the inverse of  $\eta_E$  is also positive.

The proof that  $\eta_E$  is trace-preserving is similar to that in Proposition 2.5.3 but as it is short we can show it here. The trace of  $G^\sigma(F^\beta(E))$  is  $\text{ev}(\tau)$ . If  $x \in E$  we have

$$(\text{ev}(\tau) \circ \eta_E)(x) = \eta_E(x)(\tau) = \tau(x),$$

so  $\text{ev}(\tau) \circ \eta_E = \tau$ . It follows that  $\tau \circ \eta_E^{-1} = \text{ev}(\tau)$ , so the inverse is also trace-preserving.

The proof that  $\epsilon_A$  and its inverse are positive is similar to the proof for  $\eta_E$ , except using the fact that  $A_+$  is norm-closed by Lemma A.5.3. The proof of unitality is as follows. Let  $\phi \in A^\sigma$ :

$$\epsilon_A(u)(\phi) = \phi(u) = \text{ev}(u)(\phi),$$

so  $\epsilon_A(u) = \text{ev}(u)$ , the unit of  $F^\beta(G^\sigma(A))$ . If a map is unital, then its inverse must also be, so  $\epsilon_A^{-1}$  is also unital.

In each case we know by Theorem 3.2.22 that the naturality diagrams commute with maps that are only linear and continuous, so they commute *a fortiori* for maps of Smith base-norm spaces and maps of Banach order-unit spaces. The commutativity of the diagrams to show that this is an adjoint equivalence also follows in this way.  $\square$



We define  $\text{Stat} : \mathbf{BOUS}^{\text{op}} \rightarrow \mathbf{CCL}$  to be  $B \circ G^\sigma$ . This is an equivalence of categories. In fact:

**Theorem 3.3.8.** *The functor  $\text{CAff}$  is a left adjoint to  $\text{Stat}$ . Therefore this adjunction is an adjoint equivalence.*

*Proof.* We use Theorem 0.4.1 (v) to define the adjunction by defining a unit and a counit. The counit should be  $\epsilon : \text{CAff} \circ \text{Stat} \Rightarrow \text{Id}_{\mathbf{BOUS}}$  in  $\mathbf{BOUS}^{\text{op}}$ , i.e.  $\epsilon : \text{Id}_{\mathbf{BOUS}} \Rightarrow \text{CAff} \circ \text{Stat}$  in  $\mathbf{BOUS}$ . If we temporarily use  $\epsilon'$  to refer to the counit in Theorem 3.3.7, and use the natural transformation  $\rho$  from Theorem 3.3.6, we can define

$$\epsilon = \rho G^\sigma \circ \epsilon' : \text{Id}_{\mathbf{BOUS}} \Rightarrow \text{CAff} \circ \text{Stat}.$$

By defining it in this way, it is already proven that  $\epsilon$  is well defined and natural. We can also expand the definition for  $A \in \mathbf{BOUS}$ ,  $a \in A$  and  $\phi \in \text{Stat}(A)$ :

$$\epsilon_A(a)(\phi) = \rho_{G^\sigma(A)}(\epsilon'_A(a))(\phi) = \epsilon'_A(a)(\phi) = \phi(a),$$

because  $\rho$  is just restriction to the base of a Smith base-norm space.

We define the unit as follows, for  $(E, X) \in \mathbf{CCL}$ ,  $x \in X$ , and  $a \in \text{CAff}(X)$ :

$$\eta_X(x)(a) = a(x).$$

We prove that this is defined correctly as follows.

- $\eta_X(x)$  is a state on  $\text{CAff}(X)$ :

We see that  $\eta_X(x)$  preserves addition and scalar multiplication by the pointwiseness of the definition of those operations on  $\text{CAff}(X)$ . Similarly, since the positive cone of  $\text{CAff}(X)$  is defined to be exactly those continuous affine functions taking nonnegative values at every point of  $X$ , we have that  $\eta_X(x)$  is positive. For unitality, we have  $\eta_X(x)(1) = 1(x) = 1$ .

- $\eta_X$  is an affine map  $X \rightarrow \text{Stat}(\text{CAff}(X))$ :

Let  $x, y \in X$  and  $\alpha \in [0, 1]$ . Then

$$\begin{aligned} \eta_X(\alpha x + (1 - \alpha)y) &= a(\alpha x + (1 - \alpha)y) \\ &= \alpha a(x) + (1 - \alpha)a(y) && a \text{ affine} \\ &= \alpha \eta_X(x)(a) + (1 - \alpha)\eta_X(y)(a) \\ &= (\alpha \eta_X(x) + (1 - \alpha)\eta_X(y))(a) \end{aligned}$$

- $\eta_X$  is continuous from  $X$  to the weak-\* topology on  $\text{Stat}(\text{CAff}(X))$ :

We use preservation of convergence of nets as the definition of continuity. Let  $(x_i)_{i \in I}$  be a net converging to  $x$  in  $X$ . For each  $a \in \text{CAff}(X)$  we have

$$\eta_X(x_i)(a) = a(x_i) \rightarrow a(x) = \eta_X(x)(a)$$

because  $a$  is continuous. As this is true for all  $a \in \text{CAff}(X)$ , we have that  $\eta_X(x_i) \rightarrow \eta_X(x)$  in the weak-\* topology, and therefore  $\eta_X$  is continuous.

We can show that  $\eta$  is natural as follows. The commutativity of the naturality diagram for a map  $f : X \rightarrow Y$  is equivalent to  $\eta_Y \circ f = \text{Stat}(\text{CAff}(f)) \circ \eta_X$ . If we let  $x \in X$  and  $b \in \text{CAff}(Y)$ , we have

$$\begin{aligned} (\text{Stat}(\text{CAff}(f)) \circ \eta_X)(x)(b) &= \text{Stat}(\text{CAff}(f))(\eta_X(x))(b) = \eta_X(x)(\text{CAff}(f)(b)) \\ &= \text{CAff}(f)(b)(x) = b(f(x)) = \eta_Y(f(x))(b) \\ &= (\eta_Y \circ f)(x)(b) \end{aligned}$$

The unit and counit diagrams are

$$\begin{array}{ccc} \text{CAff } X & \xleftarrow{\text{CAff } \eta_X} & \text{CAff Stat CAff } X \\ & \nwarrow \text{id}_{\text{CAff } X} & \uparrow \epsilon_{\text{CAff } X} \\ & & \text{CAff}(X) \end{array} \qquad \begin{array}{ccc} \text{Stat}(A) & \xrightarrow{\eta_{\text{Stat}(A)}} & \text{Stat CAff Stat } A \\ & \searrow \text{id}_{\text{Stat}(A)} & \downarrow \text{Stat } \epsilon_A \\ & & \text{Stat } A. \end{array}$$

To show that the left hand diagram commutes, let  $a \in \text{CAff}(X)$  and  $x \in X$  in the following:

$$\text{CAff}(\eta_X)(\epsilon_{\text{CAff}(X)}(a))(x) = \epsilon_{\text{CAff}(X)}(a)(\eta_X(x)) = \eta_X(x)(a) = a(x),$$

so  $(\text{CAff}(\eta_X) \circ \epsilon_{\text{CAff}(X)})(a) = a$  for all  $a \in \text{CAff}(X)$ , and therefore the diagram commutes.

For the right hand diagram, let  $\phi \in \text{Stat } A$  and  $a \in A$  in the following:

$$\text{Stat}(\epsilon_A)(\eta_{\text{Stat}(A)}(\phi))(a) = \eta_{\text{Stat}(A)}(\phi)(\epsilon_A(a)) = \epsilon_A(a)(\phi) = \phi(a).$$

We therefore have that  $\text{CAff} \dashv \text{Stat}$ . By composing the adjoint equivalences arising from Proposition 3.3.3 and Theorem 3.3.7, we have  $F^\beta \circ \text{Emb} \dashv B \circ G^\sigma$ . Therefore  $(\text{CAff}, \text{Stat}, \eta, \epsilon)$  forms an adjoint equivalence, and  $F^\beta \circ \text{Emb} \cong \text{CAff}$  by Lemma 0.4.4.  $\square$

We call the above adjoint equivalence *Kadison duality*, because it was Kadison who first proved that  $\epsilon_A$  was an isomorphism [64, Lemma 2.5][66, Lemma 4.3, Remark 4.4]. A more modern proof that  $\epsilon_A$  is an isomorphism can also be found in [4, Theorem II.1.8]. The name Kadison duality was first used publicly in [55].

We now have three sides of the square (3.1). We define a new functor  $\text{Stat} : \mathbf{BEMod}^{\text{op}} \rightarrow \mathbf{CCL}$ , using the same name as  $\text{Stat} : \mathbf{BOUS}^{\text{op}} \rightarrow \mathbf{CCL}$ , to be  $B(G^\sigma(A))$  for all Banach effect modules of the form  $[0, 1]_A$ . As  $[0, 1]_-$  is an equivalence, this can be extended to all of  $\mathbf{BEMod}^{\text{op}}$ , and all such extensions are naturally isomorphic. This also ensures that  $\text{Stat} \circ [0, 1]_- = B \circ G^\sigma$ .

We define  $\text{CAff}(X, [0, 1])$  for  $X$  an object of  $\mathbf{CCL}$  to be

$$\text{CAff}(X, [0, 1]) = \{a \in \text{CAff}(X) \mid \forall x \in X. 0 \leq a(x) \leq 1\}.$$

Since the order on  $\text{CAff}(X)$  is pointwise and 0 and 1 are given by constant functions, it is clear that  $\text{CAff}(X, [0, 1]) = [0, 1]_{\text{CAff}(X)}$ .

On maps  $f : (E, X) \rightarrow (F, Y)$  in  $\mathbf{CCL}$  we define

$$\text{CAff}(f, [0, 1])(b) = b \circ f.$$

This definition agrees with  $\text{CAff}(f)$ , so  $\text{CAff}(-, [0, 1]) = [0, 1]_- \circ \text{CAff}$ , and is therefore a functor. The second part of the following theorem is the author's version of [59, Theorem 6].

**Theorem 3.3.9.** *In (3.1) we have  $\text{CAff}(-, [0, 1]) \circ B \cong [0, 1]_- \circ F^\beta$  and that  $\text{CAff}(-, [0, 1])$  and  $\text{Stat}$  define an equivalence between  $\mathbf{CCL}$  and  $\mathbf{BEMod}^{\text{op}}$ .*

*Proof.* We have a natural isomorphism  $\rho : F^\beta \Rightarrow \text{CAff} \circ B$  from Theorem 3.3.6. Therefore  $[0, 1]_\rho : [0, 1]_- \circ F^\beta \Rightarrow [0, 1]_- \circ \text{CAff} \circ B = \text{CAff}(-, [0, 1]) \circ B$ , which is the isomorphism we need.

To prove that  $\text{CAff}(-, [0, 1])$  and  $\text{Stat}$  define an equivalence, we show that  $\text{CAff}(-, [0, 1]) \circ \text{Stat} \cong \text{Id}_{\mathbf{BEMod}}$  and  $\text{Stat} \circ \text{CAff}(-, [0, 1]) \cong \text{Id}_{\mathbf{CCL}}$ . We reason as follows

$$\begin{aligned} \text{CAff}(-, [0, 1]) \circ B &\cong [0, 1]_- \circ F^\beta && \Leftrightarrow \\ \text{CAff}(-, [0, 1]) \circ B \circ \text{Emb} &\cong [0, 1]_- \circ F^\beta \circ \text{Emb} && \Leftrightarrow \\ \text{CAff}(-, [0, 1]) &\cong [0, 1]_- \circ F^\beta \circ \text{Emb}, \end{aligned}$$

as  $\text{Emb}$  is an inverse for  $B$ . Similarly, we have

$$\begin{aligned} \text{Stat} \circ [0, 1]_- &= B \circ G^\sigma && \Leftrightarrow \\ \text{Stat} \circ [0, 1]_- \circ \mathcal{T} &= B \circ G^\sigma \circ \mathcal{T} && \Leftrightarrow \\ \text{Stat} &\cong B \circ G^\sigma \circ \mathcal{T}, \end{aligned}$$

as  $\mathcal{T}$  is an inverse for  $[0, 1]_-$ .

We then have

$$\begin{aligned}
 \text{Stat} \circ \text{CAff}(-, [0, 1]) &\cong B \circ G^\sigma \circ \mathcal{T} \circ [0, 1]_- \circ F^\beta \circ \text{Emb} \\
 &\cong B \circ G^\sigma \circ F^\beta \circ \text{Emb} && \text{Theorem 1.2.9} \\
 &\cong B \circ \text{Emb} && \text{Theorem 3.3.7} \\
 &\cong \text{Id}_{\mathbf{CCL}} && \text{Proposition 3.3.3.}
 \end{aligned}$$

On the other side, we have

$$\begin{aligned}
 \text{CAff}(-, [0, 1]) \circ \text{Stat} &\cong [0, 1]_- \circ F^\beta \circ \text{Emb} \circ B \circ G^\sigma \circ \mathcal{T} \\
 &\cong [0, 1]_- \circ F^\beta \circ G^\sigma \circ \mathcal{T} && \text{Proposition 3.3.3} \\
 &\cong [0, 1]_- \circ \mathcal{T} && \text{Theorem 3.3.7} \\
 &\cong \text{Id}_{\mathbf{BEMod}} && \text{Theorem 1.2.9}
 \end{aligned}$$

□

### 3.4 Compact Effect Modules and Smith Order-Unit Spaces

In this section, we reverse which kind of space has a Smith topology with respect to the previous section. Except for the subsection 3.4.1 and Ellis's theorem that an order-unit space whose unit interval is compact in a locally convex topology is the dual space of a base-norm space [34, Theorem 6], the results are original, as the notion of compact effect module does not seem to have been considered previously.

We begin with the definition of a Smith order-unit space. A *Smith order-unit space* is a quadruple  $(E, \mathcal{T}, E_+, u)$  where  $(E, \mathcal{T})$  is a locally convex topology,  $E_+$  a closed positive cone,  $(E, E_+, u)$  an order-unit space, such that  $[-u, u]$  is compact and  $\mathcal{T}$  is a Smith space topology with respect to the compact barrel  $[-u, u]$ . Unital and subunital maps of Smith order-unit spaces are simply unital and subunital maps of the underlying order-unit spaces that are continuous, and these maps define the categories **SOUS** and **SOUS**<sub>≤1</sub>.

**Proposition 3.4.1.** *If  $(A, A_+, u)$  is a partially ordered vector space with strong order unit,  $\mathcal{T}$  a locally convex topology on  $A$  in which  $[0, u]$  is compact, then  $(A, \mathcal{T}_b, A_+, \tau)$  is a Smith order-unit space, where  $\mathcal{T}_b$  is taken with respect to the compact barrel  $[-u, u]$ .*

*Proof.* We first observe that  $[-u, u]$  is compact because  $2[0, u] - u = [0, u]$  (Lemma 0.2.2) and scalar multiplication and addition are continuous and therefore map compact sets to compact sets.

Since  $[-u, u]$  is affinely isomorphic to  $[0, u]$ , it is convex, and it is balanced because  $-[-u, u] = [-u, u]$  so is absolutely convex by Lemma A.3.1. It is absorbent by the definition of a strong order unit, so  $[-u, u]$  is a compact barrel. We can therefore give  $A$  the Smith topology  $\mathcal{T}_b$  with respect to  $[-u, u]$  (Proposition 3.2.9).

We can now show that  $A_+$  is closed in this topology as follows. We have that  $A_+ \cap [-\alpha u, \alpha u] = [0, \alpha u]$  for all  $\alpha \in \mathbb{R}_{>0}$ , and therefore by Lemma 0.2.2,  $A_+ \cap \alpha[-u, u] = \alpha[0, u]$ . Since multiplication by a scalar is continuous,  $\alpha[0, u]$  is compact, and therefore closed. Therefore  $A_+$  is closed, by Lemma 3.2.10 (ii). By Lemma 3.2.3  $A_+$  is also norm-closed, and this implies  $(A, A_+, u)$  is archimedean by Lemma A.5.3. This proves it is an order-unit space, and the facts already proven show it is a Smith order-unit space.  $\square$

We can now define compact effect modules. As with compact convex sets, we deal first with the “concrete” definition, and later give an alternative definition via monads. We know that for every effect module  $A$ , there is a partially ordered vector space with strong unit  $(E, E_+, u)$  and an **EMod** isomorphism  $A \cong [0, u]_E$ . We know by Lemma A.4.1 that effect modules have an intrinsic notion of convex combination, which maps to convex combinations in  $[0, u]_E$ . We define the category **CEMod** to have objects  $(E, A)$ , where  $E$  is a locally convex space and  $A$  an effect module structure on a compact convex subset of  $E$  such that for all  $x, y \in A$  and  $\alpha \in [0, 1]$ , we have that  $\alpha x \otimes (1 - \alpha)y$ , calculated using the effect module structure, equals  $\alpha x + (1 - \alpha)y$ , calculated using the vector space structure of  $E$ . The maps in **CEMod** are effect module maps that are also continuous, and by Lemma A.4.1 they are affine.

We first prove a lemma about elements of **CEMod**.

**Lemma 3.4.2.** *Let  $(E, A) \in \mathbf{CEMod}$ . If  $a, b \in A$  such that  $a \perp b$ , we have*

$$a \otimes b = a + b,$$

*where  $+$  is the vector space addition for  $E$ . We also have that  $a^\perp = 1 - a$ ,  $1$  being the unit element of  $A$ . If  $0_E = 0_A$ , then if  $\alpha \in [0, 1]$ ,  $\alpha \cdot_A a = \alpha \cdot_E a$ , where the subscript on the  $\cdot$  and  $0$  indicates which structure it refers to.*

*Proof.* For the additive part, we reason as follows, with all scalar multiplica-

tions being in  $A$ , not  $E$ .

$$\begin{aligned}
a \otimes b &= \frac{1}{2}(a \otimes b) \otimes \frac{1}{2}(a \otimes b) \\
&= \frac{1}{2}(a \otimes b) + \frac{1}{2}(a \otimes b) && \text{convex combinations} \\
&= \left(\frac{1}{2}a \otimes \frac{1}{2}b\right) + \left(\frac{1}{2}a \otimes \frac{1}{2}b\right) && \text{effect module axiom} \\
&= \left(\frac{1}{2}a + \frac{1}{2}b\right) + \left(\frac{1}{2}a + \frac{1}{2}b\right) && \text{convex combinations} \\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \left(\frac{1}{2}b + \frac{1}{2}b\right) \\
&= \left(\frac{1}{2}a \otimes \frac{1}{2}a\right) + \left(\frac{1}{2}b \otimes \frac{1}{2}b\right) && \text{convex combinations} \\
&= a + b,
\end{aligned}$$

the last step using the effect module axioms.

We now have that because  $a \otimes a^\perp = 1$ ,  $a + a^\perp = 1$ , and so  $a^\perp = 1 - a$ .

We now assume that the zero of  $A$  is the same element as the zero of  $E$ , or in our notation,  $0_A = 0_E$ . Then

$$\begin{aligned}
\alpha \cdot_A a &= \alpha \cdot_A a \otimes 0_A && \text{effect algebra axiom} \\
&= \alpha \cdot_A a \otimes (1 - \alpha) \cdot_A 0_A && \text{Lemma A.4.2} \\
&= \alpha \cdot_E a + (1 - \alpha) \cdot_E 0_A && \text{convex combinations} \\
&= \alpha \cdot_E a + (1 - \alpha) \cdot_E 0_E && 0_A = 0_E \\
&= \alpha \cdot_E a.
\end{aligned}$$

□

For each order-unit space  $(A, A_+, u)$ , we have seen that we have an effect module  $[0, 1]_A$ , and this in fact defines a functor  $\mathbf{OUS} \rightarrow \mathbf{EMod}$ . We can now deal with the analogue of this for Smith order-unit spaces and compact effect modules.

We can see that if  $(A, \mathcal{T}, A_+, u)$  is a Smith order-unit space, then  $(A, [0, 1]_A)$  is an object of  $\mathbf{CEMod}$ . Additionally, if  $f : A \rightarrow B$  is a unital morphism of Smith spaces, we already know  $f|_{[0, 1]_A}$  is a morphism of effect modules, and it is a  $\mathbf{CEMod}$  map because it is continuous. As this is simply restriction of functions,  $[0, 1]_-$  is a functor  $\mathbf{SOUS} \rightarrow \mathbf{CEMod}$ .

**Theorem 3.4.3.** *The functor  $[0, 1]_- : \mathbf{SOUS} \rightarrow \mathbf{CEMod}$  is an equivalence.*

*Proof.* In the following, let  $(A, \mathcal{T}, A_+, u)$  and  $(B, \mathcal{S}, B_+, v)$  be Smith order-unit spaces.

The functor  $[0, 1]_-$  is faithful because for any order-unit space,  $[0, 1]_A$  spans  $A$ , so if  $f|_{[0, 1]_A} = g|_{[0, 1]_A}$ , then  $f = g$  by linearity.

Now let  $g : [0, 1]_A \rightarrow [0, 1]_B$  be a continuous effect module homomorphism. We know by the fullness of  $[0, 1]_- : \mathbf{poVectu} \rightarrow \mathbf{EMod}$  that there is an  $f : A \rightarrow B$  that is linear, positive and unital such that  $[0, 1]_f = g$ . Therefore  $f|_{[0, 1]_A}$  is continuous. However, we need to show that  $f$  is continuous. We first show that  $f|_{[0, 2]_A}$  is continuous.

Let  $V \subseteq B$  be an open set. Since  $B$  is a topological vector space  $\frac{1}{2}B$  is also open. By continuity of  $g$ , we have that  $f^{-1}(\frac{1}{2}V) \cap [0, u] = U' \cap [0, u]$  for some  $U'$  an open subset of  $A$ . Multiplying both sides by 2, we get

$$2f^{-1}\left(\frac{1}{2}V\right) \cap [0, 2u] = 2U' \cap [0, 2u],$$

and using the linearity of  $f$  this shows

$$f^{-1}(V) \cap [0, 2u] = 2U' \cap [0, 2u].$$

Since  $2U'$  is an open subset of  $A$ , this implies that  $f|_{[0, 2]_A}$  is continuous. From here, we can show that  $f|_{[-1, 1]_A}$  is continuous.

So again, let  $V \subseteq B$  be an open set. We see that  $V + f(u)$  is also open, as  $B$  is a topological vector space. We therefore have that  $f^{-1}(V + f(u)) \cap [0, 2u] = U' \cap [0, 2u]$  for some open  $U' \subseteq A$ . We can then subtract  $u$  from both sides and get

$$\begin{aligned} (f^{-1}(V + f(u)) \cap [0, 2u]) - u &= (U' \cap [0, 2u]) - u && \Leftrightarrow \\ (f^{-1}(V) + u - u) \cap [-u, u] &= (U' - u) \cap [-u, u] && \Leftrightarrow \\ f^{-1}(V) \cap [-u, u] &= (U' - u) \cap [-u, u]. \end{aligned}$$

We have that  $U' - u$  is an open subset of  $A$ , so  $f|_{[-1, 1]_A}$  is continuous. We then apply Proposition 3.2.15 to conclude that  $f$  is continuous, and therefore a map in  $\mathbf{SOUS}$ , proving the fullness.

We now move on to the final stage, proving that  $[0, 1]_-$  is essentially surjective. Let  $(E, A) \in \mathbf{CEMod}$ . We need to find  $(A', \mathcal{T}, A'_+, u) \in \mathbf{SOUS}$  and a continuous effect module isomorphism  $i : A \rightarrow [0, 1]_{A'}$ . For purposes of disambiguation, we will at first use  $0_A$  to refer to the 0 element of  $A$  and  $0_E$  to that

of  $E$ . We can then redefine  $A$  to be  $A - 0_A$  and make  $- - 0_A$  an isomorphism  $(E, A) \cong (E, A - 0_A)$ , so that  $0_A = 0_E$ , which we now refer to as  $0$  again.

Define  $A' = \text{span}(A)$ ,  $A'_+ = \bigcup_{\alpha \in \mathbb{R}_{>0}} \alpha A$  and  $u = 1_A$ . We can show that  $[0, 1]_{A'} = A$  right away:

$$[0, 1]_{A'} = \{a \in \text{span}(A) \mid a \in A'_+ \text{ and } u - a \in A'_+\} = \{a \in A'_+ \mid u - a \in A'_+\},$$

as  $A'_+ \subseteq \text{span}(A)$  by definition. Suppose that  $a$  is an element of this set, *i.e.* that  $a = \lambda a'$  where  $a' \in A$  and  $\lambda \in \mathbb{R}_{>0}$ , and  $u - a = \mu a''$  where  $a'' \in A$  and  $\mu \in \mathbb{R}_{>0}$ , which is to say that  $a = u - \mu a''$ . Eliminating  $a$ , we get that  $\lambda a' = u - \mu a''$ , or  $\lambda a' + \mu a'' = u$ . We know that  $\lambda + \mu > 0$  and we can take  $n = \lceil \lambda + \mu \rceil \geq 1$ . Then  $\frac{\lambda}{n} + \frac{\mu}{n} \leq 1$ , so the equation

$$\frac{\lambda}{n} a' \odot \frac{\mu}{n} a'' = \frac{1}{n} u$$

holds in  $A$  (Lemma A.4.1). We can then add this equation to itself  $n$  times and rearrange the terms using commutativity and associativity to get

$$\left( \frac{\lambda}{n} a' \odot \cdots \odot \frac{\lambda}{n} a' \right) \odot \left( \frac{\mu}{n} a'' \odot \frac{\mu}{n} a'' \right) = u,$$

and we therefore have that these repeated additions are defined as elements of  $A$ . We can apply Lemma 3.4.2 to conclude that

$$\frac{\lambda}{n} a' \odot \cdots \odot \frac{\lambda}{n} a' = \lambda a'$$

by replacing the effect module operations by the vector space ones, and we therefore have  $a = \lambda a' \in A$ . This shows that  $[0, 1]_A \subseteq A$ . Now if  $a \in A$ , we have that  $a \in A_+$ , with  $\lambda = 1$ , and  $a \in u - A_+$  because  $a = 1 - a^\perp$  by Lemma 3.4.2, so  $A \subseteq [0, 1]_A$ . We now show that  $(A', A'_+, u)$  is a partially ordered vector space with strong order unit.

- $A'_+$  is a cone:

For closure under addition, let  $a, b \in A'_+$ , *i.e.*  $a = \lambda a'$  and  $b = \mu b'$  for  $a', b' \in A$ ,  $\lambda, \mu \in \mathbb{R}_{>0}$ . By Lemma A.4.1, we have

$$\frac{\lambda}{\lambda + \mu} a' \odot \frac{\mu}{\lambda + \mu} b' \in A,$$

and

$$\begin{aligned} a + b &= \lambda a' + \mu b' = (\lambda + \mu) \left( \frac{\lambda}{\lambda + \mu} a' + \frac{\mu}{\lambda + \mu} b' \right) \\ &= (\lambda + \mu) \left( \frac{\lambda}{\lambda + \mu} a' \odot \frac{\mu}{\lambda + \mu} b' \right), \end{aligned}$$



by preservation of convex combinations. Therefore  $a + b \in A'_+$ .

We have that  $0 \in A \subseteq A'_+$  directly. To show that  $A'_+$  is closed under positive scalar multiplication, we separate into two cases. Let  $a = \lambda a'$  with  $a \in A$  as before. If  $\mu \in \mathbb{R}_{>0}$ , then  $\mu a = \mu \lambda a'$ , and since  $\mu \lambda > 0$  we have that  $\mu a \in A'_+$ . The other case is when  $\mu = 0$ . In this case,  $\mu a = 0$ , which is in  $A'_+$  as we already showed.

- $u$  is a strong order unit:

We show this in two steps using Lemma A.5.1. We first show that  $A'_+ - A'_+ = A'$ , as follows. Since  $A'$  is the span of  $A$ , we have that every  $a \in A'$  can be expressed as  $\sum_{i \in I} \alpha_i a_i$ , where  $\alpha_i \in \mathbb{R} \setminus 0^3$ ,  $a_i \in A$ , and  $I$  is a finite set. We then define

$$I_+ = \{i \in I \mid \alpha_i > 0\} \quad I_- = \{i \in I \mid \alpha_i < 0\},$$

and we have  $I_+ \cup I_- = I$ . Therefore

$$a = \sum_{i \in I_+} \alpha_i a_i - \sum_{i \in I_-} (-\alpha_i) a_i,$$

and this expresses  $a$  as a difference of two elements of  $A'_+$ .

We can then see that for  $\alpha \in \mathbb{R}_{>0}$ ,  $\alpha[0, u] = [0, \alpha u]$  as follows:

$$\begin{aligned} x \in \alpha[0, u] &\Leftrightarrow \alpha^{-1}x \in [0, u] \\ &\Leftrightarrow \alpha^{-1}x \in A'_+ \text{ and } \alpha^{-1}x \in u - A'_+ \\ &\Leftrightarrow \alpha^{-1}x \in A'_+ \text{ and } u - \alpha^{-1}x \in A'_+ \\ &\Leftrightarrow x \in A'_+ \text{ and } \alpha u - x \in A'_+ \quad A'_+ \text{ a cone} \\ &\Leftrightarrow x \in [0, \alpha u]. \end{aligned}$$

Therefore

$$\bigcup_{n \in \mathbb{N}} [0, nu] = \bigcup_{\alpha \in \mathbb{R}_{>0}} [0, \alpha u] = \bigcup_{\alpha \in \mathbb{R}_{>0}} \alpha[0, u] = A'_+,$$

so  $u$  is a strong order unit.

We then use the compactness of  $A = [0, u]$  to apply Proposition 3.4.1, defining  $\mathcal{T}$  to be the Smithification of the original (subspace) topology on  $A$ , to get that  $(A', \mathcal{T}, A'_+, u)$  is a Smith order-unit space.  $\square$

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<sup>3</sup>We can exclude any zero terms without affecting the value of the sum.

Now we define  $G^\beta$ . On a Smith order-unit space  $(A, \mathcal{T}, A_+, u)$  we define

$$G^\beta(A) = (A^\beta, A_+^\beta, \text{ev}(u)),$$

and for a continuous unital or subunital map  $f : (A, \mathcal{T}, A_+, u) \rightarrow (B, \mathcal{S}, B_+, v)$  and  $\phi \in B^\beta$

$$G^\beta(f)(\phi) = \phi \circ f$$

**Proposition 3.4.4.**  $G^\beta$  is defines a functor  $\mathbf{SOUS}_{\leq 1}^{\text{op}} \rightarrow \mathbf{BBNS}_{\leq 1}$  and a functor  $\mathbf{SOUS}^{\text{op}} \rightarrow \mathbf{BBNS}$ .

*Proof.* We first show that  $G^\beta(A, \mathcal{T}, A_+, u) = (A^\beta, A_+^\beta, \text{ev}(u))$  is a Banach base-norm space. In aid of this, we define:

$$\begin{aligned} F &= A_+^\beta - A_+^\beta \subseteq A^\beta \\ B^{\leq 1} &= A_+^\beta \cap \text{ev}(u)^{-1}((-\infty, 1]) = \{\phi \in A^\beta \mid \phi(u) \leq 1 \text{ and } \forall a \in A_+. \phi(a) \geq 0\} \\ V &= \text{co}(B^{\leq 1} \cup -B^{\leq 1}). \end{aligned}$$

We already know that  $A^\beta$  is a Banach space with unit ball  $[-u, u]^\circ$  by Proposition 3.2.13. We show that  $B^{\leq 1} = A_+^\beta \cap [-u, u]^\circ$  as follows. If we start with  $\phi \in [-u, u]^\circ$ , we can conclude from  $u \in [-u, u]$  that  $\text{ev}(u)(\phi) = \phi(u) \leq 1$ , so  $\phi \in \text{ev}(u)^{-1}((-\infty, 1])$ . Therefore  $A_+^\beta \cap [-u, u]^\circ \subseteq B^{\leq 1}$ . For the other direction, observe that if  $\phi \in A_+^\beta$ , then it preserves positive elements as a map  $\phi : A \rightarrow \mathbb{R}$ , so is a monotone map. Therefore if  $\phi \in A_+^\beta \cap \text{ev}(u)^{-1}((-\infty, 1])$ , and  $a \in [-u, u]$ , we have  $\phi(a) \leq \phi(u) = \text{ev}(u)(\phi) \leq 1$ , so  $\phi \in [-u, u]^\circ$ .

As  $A_+^\beta$  and  $[-u, u]^\circ$  are closed in  $A^\beta$ ,  $B^{\leq 1}$  is closed, and therefore complete, and so is  $\sigma$ -convex (by Lemma 0.1.19). We can therefore apply Lemma 2.2.15 to deduce that  $(F, \|\cdot\|_V)$  is a Banach space. Consider the inclusion mapping  $i : F \rightarrow A^\beta$ , which is a contraction as  $i(V) \subseteq [-u, u]^\circ$  because  $B^{\leq 1} \subseteq [-u, u]^\circ$  and  $[-u, u]^\circ$  is absolutely convex. The map  $i^\sigma : A^{\beta\sigma} \rightarrow F^\sigma$  exists as a map of Smith spaces. We aim to show that this is a **Smith**<sub>1</sub> isomorphism, from which we could conclude that  $F = A^\beta$ .

First consider  $V$  as a subset of  $A^\beta$ , in its pairing with  $A$ . We can show that  $V^\circ = [-u, u]$  as follows.

$$\begin{aligned} V^\circ &= \text{co}(B^{\leq 1} \cup -B^{\leq 1})^\circ \\ &= \text{absco}(B^{\leq 1})^\circ \\ &= B^{\leq 1|\circ|} && \text{Lemma 0.3.11 (iv)} \\ &= \{a \in A \mid \forall \phi \in B^{\leq 1}. -1 \leq \phi(a) \leq 1\} = X \end{aligned}$$

We now show that this set  $X$  that we have just defined is equal to

$$Y = \{a \in A \mid \forall \phi \in A_+^\beta. -\phi(u) \leq a \leq \phi(u)\}.$$

- $X \subseteq Y$ :

Let  $a \in X$  and suppose that  $\phi \in A_+^\beta$ . Then if  $\phi(u) = 0$ , by Lemma A.5.4  $\phi = 0$ , so  $-\phi(u) \leq \phi(a) \leq \phi(u)$  because they are all zero. If  $\phi(u) \neq 0$ , then  $\phi(u) > 0$ . Let  $\alpha = \phi(u)$ . The map  $\alpha^{-1}\phi \in B^{\leq 1}$ , and therefore

$$-1 \leq \alpha^{-1}\phi(a) \leq 1.$$

Multiplying through by  $\alpha$  and substituting it for its definition we get

$$-\phi(u) \leq \phi(a) \leq \phi(u)$$

as required.

- $Y \subseteq X$ : If  $a \in Y$ , and  $\phi \in B^{\leq 1}$ , then  $\phi(u) \leq 1$  by definition. Therefore

$$-1 \leq -\phi(u) \leq \phi(a) \leq \phi(u) \leq 1.$$

So far we have shown that  $V^\circ = Y$ . Now

$$\begin{aligned} Y &= \{a \in A. \forall \phi \in A_+^\beta. -\phi(u) \leq \phi(a) \leq \phi(u)\} \\ &= \{a \in A. (\forall \phi \in A_+^\beta. \phi(u+a) \geq 0) \text{ and} \\ &\quad (\forall \phi \in A_+^\beta. \phi(u-a) \geq 0)\} \\ &= \{a \in A. u+a \in A_+ \text{ and } u-a \in A_+\} \quad \text{Lemma 0.3.15} \\ &= [-u, u]. \end{aligned}$$

Therefore  $V^{\circ\circ} = [-u, u]^\circ = U$ . Therefore  $U$  is the  $\sigma(A^\beta, A)$ -closure of  $V$  (Corollary 0.3.10), and therefore the  $\sigma(A^\beta, A^{\beta\sigma})$ -closure of  $V$  (Proposition 3.2.17), which by Proposition 0.3.4 is in fact the  $\|\cdot\|_U$ -closure. So if  $\phi \in A^\beta$ , there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $\alpha^{-1}\phi \in U$ . There is therefore a sequence  $(\psi_i)_{i \in \mathbb{N}}$  in  $V$ , and therefore in  $F$ , converging in  $\|\cdot\|_U$  to  $\alpha^{-1}\phi$ . Therefore  $\alpha\psi_i \rightarrow \phi$ , and so  $F$  is  $\|\cdot\|_U$ -dense in  $A^\beta$ . As the inclusion map  $i$  is also injective, by Proposition 3.2.25  $i^\sigma$  is injective with dense image.

If we consider the pairings between  $F$  and  $F^\sigma$  and  $A^\beta$  and  $A^{\beta\sigma}$ , the map  $i : F \rightarrow A^\beta$  has adjoint  $i^\sigma : A^{\beta\sigma} \rightarrow F^\sigma$ . Considering the polars with respect to these dualities, we have

$$(i^\sigma)^{-1}(V^\circ) = i(V)^\circ = U^\circ$$

by Lemma 0.3.14 and the fact that  $U = \text{cl}(V)$ . Therefore

$$i^\sigma(U^\circ) = i^\sigma((i^\sigma)^{-1}(V^\circ)) = i^\sigma(A^{\beta\sigma}) \cap V^\circ.$$

As  $U^\circ$  is compact, being the unit ball of  $A^{\beta\sigma}$ ,  $i^\sigma(U^\circ)$  is compact, and therefore closed. We have therefore shown that  $i^\sigma(A^{\beta\sigma}) \cap V^\circ$  is closed. By multiplying by  $\alpha$  and using the fact that  $i^\sigma(A^{\beta\sigma})$  is a subspace, we see that  $i^\sigma(A^{\beta\sigma}) \cap \alpha V^\circ$  is closed. By Lemma 3.2.10,  $i^\sigma(A^{\beta\sigma})$  is closed, and as we already showed it was dense in  $F^\sigma$ , we have that  $i^\sigma$  is surjective. Therefore  $i^\sigma$  is a continuous linear bijection of Smith spaces, so therefore is an isomorphism of Smith spaces by Corollary 3.2.19.

All together, we have seen that  $i^\sigma$  is an isomorphism in **Smith**<sub>1</sub>, so by Corollary 3.2.23  $i$  is an isomorphism in **Ban**<sub>1</sub>. Therefore  $A^\beta = A_+^\beta - A_+^\beta$ . The map  $\text{ev}(u) : A^\beta \rightarrow \mathbb{R}$  is positive because if  $\phi \in A_+^\beta$ ,  $\text{ev}(u)(\phi) = \phi(u) \geq 0$  because  $u \in A_+$ . Suppose  $A^\beta \neq \{0\}$ , and therefore  $A_+^\beta \neq \{0\}$ . Then there exists  $\phi \in A_+^\beta \neq 0$ . If  $\text{ev}(u)(\phi) = \phi(u) = 0$ , Lemma A.5.4 implies  $\phi = 0$ , contradicting the assumption. We now define  $B = A_+^\beta \cap \text{ev}(u)^{-1}(1)$ .

- $\text{absco}(B) \subseteq V$ :

Recall that  $V = \text{co}(-B^{\leq 1} \cup B^{\leq 1})$ , and  $B^{\leq 1} = A_+^\beta \cap \text{ev}(u)^{-1}((-\infty, 1])$ . As  $B^{\leq 1}$  contains zero,  $V$  is nonempty, and it is convex by definition. If  $\alpha\phi_+ - (1 - \alpha)\phi_-$  is an element of  $V$ , i.e.  $\alpha \in [0, 1]$ ,  $\phi_+, \phi_- \in B^{\leq 1}$ , then

$$-(\alpha\phi_+ - (1 - \alpha)\phi_-) = (1 - \alpha)\phi_- - \alpha\phi_+,$$

which is also an element of  $V$ . Therefore  $V$  is balanced, and so absolutely convex by Lemma A.3.1. Since  $B \subseteq B^{\leq 1}$ ,  $\text{absco}(B) \subseteq \text{absco}(B^{\leq 1}) \subseteq V$ .

- $V \subseteq \text{absco}(B)$ :

As  $\text{absco}(B)$  is balanced,  $B^{\leq 1} \subseteq \text{absco}(B)$  iff  $-B^{\leq 1} \subseteq \text{absco}(B)$ , and as it is convex each of these implies  $V \subseteq \text{absco}(B)$ . Therefore we reduce to showing that  $B^{\leq 1} \subseteq \text{absco}(B)$ . Let  $\phi \in B^{\leq 1}$ . If  $\phi(u) = 0$ , we have  $\phi = 0$  and so  $\phi \in \text{absco}(B)$  (Lemma A.5.4). If  $\phi(u) \neq 0$ , and therefore  $\phi(u) \in (0, 1]$ , define  $\alpha = \phi(u)$ . We have that  $\alpha^{-1}\phi$  is in  $A_+^\beta$  and maps  $u$  to 1, hence is an element of  $B$ . Therefore  $\phi = (1 - \alpha)0 + \alpha(\alpha^{-1}\phi)$  is an element of  $\text{absco}(B)$ .

As we know that  $V$  is radially compact, we have  $\text{absco}(B)$  is radially compact, so we have a pre-base-norm space. We also showed already that  $A_+^\beta$  is complete, and therefore closed in  $\|\cdot\|_V$ , so we have a base-norm space. Since

$F = A^\beta$  is complete in  $\|\cdot\|_V$ , we have a Banach base-norm space. We have finished with the object part of the functor.

Now we show that for  $f : (A, \mathcal{T}, A_+, u) \rightarrow (B, \mathcal{S}, B_+, v)$  a continuous sub-unital map  $G^\beta(f)$  is a trace-reducing map, which is trace-preserving if  $f$  is unital. Since  $G^\beta(f) = f^\beta$ , it is a linear map (Proposition 3.2.20), and it is positive by the positivity argument in Proposition 2.5.2. Additionally, the proofs that  $G^\beta(f)$  is trace-reducing or trace-preserving when  $f$  is subunital or unital respectively are similar to the proofs for  $G$  in Proposition 2.5.2. Preservation of identity maps and composition follows from the identity map laws and associativity of composition of continuous linear maps in the usual way.  $\square$

We can also define  $F^\sigma : \mathbf{BBNS} \rightarrow \mathbf{SOUS}^{\text{op}}$ . We give it almost the same definitions as  $F : \mathbf{BBNS} \rightarrow \mathbf{BOUS}^{\text{op}}$ :

$$\begin{aligned} F^\sigma(E, E_+, \tau) &= (E^\sigma, \sigma(E^\sigma, E)_{[-\tau, \tau]}, E_+^\sigma, \tau) \\ F^\sigma(f)(b) &= b \circ f, \end{aligned}$$

where  $f : (E, E_+, \tau) \rightarrow (F, F_+, \sigma)$  is a trace-reducing map and  $b \in F^\sigma$ .

**Theorem 3.4.5.**  *$F^\sigma$  is a functor from  $\mathbf{BBNS} \rightarrow \mathbf{SOUS}^{\text{op}}$ . The restriction of the adjoint equivalence defined by  $\sigma$  and  $\beta$  makes an adjoint equivalence  $\mathbf{BBNS} \simeq \mathbf{SOUS}^{\text{op}}$  and  $\mathbf{BBNS}_{\leq 1} \simeq \mathbf{SOUS}_{\leq 1}^{\text{op}}$ .*

*Proof.* We have that  $F^\sigma(E, E_+, \tau)$  is a Banach order-unit space by Proposition 2.4.17, and that  $[-\tau, \tau]$  is the unit ball in the usual dual norm. Therefore  $\sigma(E^\sigma, E)_{[-\tau, \tau]}$  is a Smith topology on  $F^\sigma(E, E_+, \tau)$  (Proposition 3.2.9). All we need to show that  $F^\sigma(E, E_+, \tau)$  is a Smith order-unit space is to show that  $E_+$  is closed. We know that it is closed in  $\sigma(E^\sigma, E)$  because it is a dual cone, and therefore a polar. Therefore it is closed in the finer  $\sigma(E^\sigma, E)_{[-\tau, \tau]}$ -topology.

In Proposition 2.5.2 it is shown that if  $f : (E, E_+, \tau) \rightarrow (F, F_+, \sigma)$  is trace-reducing, then  $F(f)$  is subunital, and if  $f$  is trace-preserving,  $F(f)$  is unital, as well as  $F$  preserving identity maps and composition. Therefore to show  $F$  is functor we only need to show that if  $f$  is trace-reducing,  $F^\sigma(f)$  is continuous in the Smith topologies. This has already been shown for  $f$  a bounded linear map in Proposition 3.2.21, so we only need to use Proposition 2.2.12, that trace-reducing implies bounded.

We consider the usual

$$\begin{aligned} \eta_E : E &\rightarrow G^\beta(F^\sigma(E)) & \epsilon_A : A &\rightarrow F^\sigma(G^\beta(A)) \\ \eta_E(x)(a) &= a(x) & \epsilon_A(a)(\phi) &= \phi(a). \end{aligned}$$

By Theorem 3.2.22 these are linear homeomorphisms of the underlying topological vector spaces, and the diagrams required for an adjoint equivalence commute. Therefore we only need to show that  $\eta_E$  is an isomorphism in **BBNS** and  $\epsilon_A$  an isomorphism in **SOUS**.

If  $x \in E_+$ , then for all  $a \in E_+^\sigma$ ,  $\eta_E(x)(a) = a(x) \geq 0$ , so  $\eta_E(x) \in E_+^{\sigma\beta}$ . If, on the other hand, if  $\phi \in G^\beta(F^\sigma(E))_+$ , i.e.  $E_+^{\sigma\beta}$ , then by bijectivity of  $\eta_E$  there exists an  $x \in E_+$  such that  $\eta_E(x) = \phi$ . So for all  $a \in E_+^\sigma$  there  $a(x) = \eta_E(x)(a) \geq 0$ , so  $x$  is in the dual cone of  $E^\sigma$  under the duality  $(E, E^\sigma, \langle -, - \rangle)$ . As  $E_+$  is closed, by the definition of a base-norm space, Lemma 0.3.15 shows that  $x \in E_+$ . Therefore  $\eta_E$  and its inverse are both positive.

The proof that  $\epsilon_A$  and its inverse are positive is similar, using the fact that  $A_+$  is required to be closed in the Smith topology as part of the definition of a Smith order-unit space.

The proofs of trace-preservation and unitality are similar to Theorem 3.3.7, so are omitted.  $\square$

The previous theorem can be considered to be a categorical version of Ellis's result [34, Theorem 6] that an order-unit space  $(A, A_+, u)$  is the dual of a base-norm space iff  $A$  can be equipped with a locally convex topology in which  $[0, 1]_A$  is compact.

### 3.4.1 Relationship to Convex Sets

In this subsection we show that **BAff** produces a Smith order-unit space and that **CBConv**, and therefore **BOUS**, forms a reflective subcategory of both  $\mathcal{EM}(\mathcal{D})$  and  $\mathcal{EM}(\mathcal{D}_\infty)$  via the comparison functors (2.2) and (2.4). The topology used to make **BAff**( $X$ ) Smith, a variation of the bounded weak-\* topology, was first defined in [91, Theorem 3], in the more general case where  $X$  is a convex prestructure, for the purpose of producing, for each  $X$ , a unique base-norm space whose dual space was isomorphic to **BAff**( $X$ ) (a theorem strongly related to Corollary 3.4.12). There is also Pumplün's result in [98, Theorem 3.3], although "base-norm space" in that reference refers to a pre-base-norm space, so that result is not quite the same as Corollary (3.4.12).

To redefine **BAff**, we first define  $\eta_X : X \rightarrow \text{Stat}(\text{BAff}(X))$  as

$$\eta_X(x)(a) = a(x).$$

**Lemma 3.4.6.** *For all  $(X, \alpha_X)$  in  $\mathcal{EM}(\mathcal{D})$  and  $x \in X$ ,  $\eta_X(x)$  is a state on **BAff**( $X$ ).*

*Proof.* To show that  $\eta_X(x)$  is linear, consider  $a, b \in \text{BAff}(X)$ . Then

$$\eta_X(x)(a + b) = (a + b)(x) = a(x) + b(x) = \eta_X(x)(a) + \eta_X(x)(b).$$

And if  $\alpha \in \mathbb{R}$ ,

$$\eta_X(x)(\alpha a) = (\alpha a)(x) = \alpha a(x) = \alpha \eta_X(x)(a).$$

For positivity, let  $a \in \text{BAff}(X)_+$ . Then

$$\eta_X(x)(a) = a(x) \geq 0,$$

by the definition of  $\text{BAff}(X)_+$ . For unitality,  $\eta_X(x)(u) = u(x) = 1$  by the definition of  $u$ .  $\square$

We have that  $\eta_X(X) \subseteq \text{BAff}(X)^*$  because all states are continuous (Proposition 1.2.8). Therefore  $\text{span}(\eta_X(X))$  is a subspace of  $\text{BAff}(X)^*$ .

**Lemma 3.4.7.** *The set  $\eta_X(X) \subseteq \text{BAff}(X)^*$  separates the points of  $\text{BAff}(X)$ , therefore the topology  $\sigma(\text{BAff}(X), \text{span}(\eta_X(X)))$  is a Hausdorff locally convex topology.*

*Proof.* Let  $a, b \in \text{BAff}(X)$ . If for all  $\phi \in \text{span}(\eta_X(X))$ , then for all  $x \in X$  we have  $\eta_X(x)(a) = \eta_X(x)(b)$ , i.e.  $a(x) = b(x)$ , so  $a = b$ . Therefore  $\sigma(\text{BAff}(X), \eta_X(X)) = \sigma(\text{BAff}(X), \text{span}(\eta_X(X)))$  is a Hausdorff locally convex topology.  $\square$

We can now show that  $\text{BAff}(X)$  is a Smith space.

**Proposition 3.4.8.** *The interval  $[0, 1]_{\text{BAff}(X)}$  is  $\sigma(\text{BAff}(X), \eta_X(X))$  compact, so  $\text{BAff}(X)$  is a Smith space.*

*Proof.* Each element  $a \in [0, 1]_{\text{BAff}(X)}$  is a function from  $X \rightarrow [0, 1]$ , or an element of  $[0, 1]^X$ . By Tychonoff's theorem,  $[0, 1]^X$  is a compact Hausdorff space, so if we show that  $[0, 1]_{\text{BAff}(X)}$  is closed in  $[0, 1]^X$ , with its standard topology, and that the topology agrees with the  $\sigma(\text{BAff}(X), \text{span}(\eta_X(X)))$  topology, we have shown that  $[0, 1]_{\text{BAff}(X)}$  is compact.

First we show that every  $\sigma(\text{BAff}(X), \eta_X(X))$  neighbourhood of a point  $a \in [0, 1]_{\text{BAff}(X)}$  is a neighbourhood in subspace topology from  $[0, 1]^X$ . A base of neighbourhoods is defined by sets defined as follows. Given a finite set  $I$ , and finite sequences  $(x_i)_{i \in I}$ ,  $x_i \in X$ , and  $(\epsilon_i)_{i \in I}$ ,  $\epsilon_i \in \mathbb{R}_{>0}$ , we take

$$N_{a, (x_i), (\epsilon_i)} = \bigcap_{i \in I} N_{a, x_i, \epsilon_i} = \bigcap_{i \in I} \{b \in \text{BAff}(X) \mid |a(x_i) - b(x_i)| < \epsilon_i\}.$$

Sets of this form make up a base of  $a$ -neighbourhoods in  $\sigma(\text{BAff}(X), \eta_X(X))$ . Now,  $|a(x_i) - b(x_i)| < \epsilon_i$  iff  $b(x_i) \in (a(x_i) - \epsilon_i, a(x_i) + \epsilon_i)$ , and

$$\{b \in [0, 1]^X \mid b(x_i) \in (a(x_i) - \epsilon_i, a(x_i) + \epsilon_i)\} = \pi_{x_i}^{-1}((a(x_i) - \epsilon_i, a(x_i) + \epsilon_i))$$

is an open set in the product topology, so we have shown that the product topology is finer than  $\sigma(\text{BAff}(X), \eta_X(X))$ .

Now, let  $I$  be a finite set,  $(U_i)_{i \in I}$  a finite sequence of open subsets of  $\mathbb{R}$ , and  $(x_i)_{i \in I}$  a finite sequence of elements of  $X$  such that  $a(x_i) \in U_i$ . Then

$$\bigcap_{i \in I} \pi_{x_i}^{-1}(U_i)$$

is an open neighbourhood of  $a$  in the product topology, and the family of sets of this form make a neighbourhood base for  $a$ . As each  $U_i$  is open, we can pick an  $(\epsilon_i)_{i \in I}$  such that  $(a(x) - \epsilon_i, a(x) + \epsilon_i) \subseteq U_i$  for all  $i \in I$ . Then  $N_{a, x_i, \epsilon_i} \subseteq \pi_{x_i}^{-1}(U_i)$ , so the  $\sigma(\text{BAff}(X), \eta_X(X))$  topology is finer than the product topology. Combining this with the previous paragraph proves that they are both the same.

We can now move on to showing that  $[0, 1]_{\text{BAff}(X)} \subseteq [0, 1]^X$  is closed. Let  $(a_i)_{i \in I}$  be a net in  $[0, 1]_{\text{BAff}(X)}$  converging to  $a \in [0, 1]^X$ . Since the function  $a \in [0, 1]^X$  is bounded, we only need to show that it is  $\mathcal{D}$ -affine. Let  $\phi \in \mathcal{D}(X)$ , and we want to show that  $\sum_{x \in X} \phi(x)a(x) = a(\alpha_X(\phi))$ . For all  $i \in I$ , we have  $\Phi(a_i) = \sum_{x \in X} \phi(x)a_i(x) = a_i(\alpha_X(\phi))$ . Since convergence in the product topology is pointwise, we have that  $a_i(\alpha_X(\phi)) \rightarrow a(\alpha_X(\phi))$ . Since the sum is finite, and addition and scalar multiplication are continuous,  $\Phi(a_i) \rightarrow \Phi(a)$ . As the topology is Hausdorff, we have that  $\Phi(a) = a(\alpha_X(\phi))$ , i.e.  $a$  is  $\mathcal{D}$ -affine.

We then apply Proposition 3.4.1 (using the fact that  $\sigma(\text{BAff}(X), \eta_X(X))$  is a locally convex topology from Lemma 3.4.7) to conclude that

$$(\text{BAff}(X), \sigma(\text{BAff}(X), \eta_X(X))_{[-u, u]}, \text{BAff}(X)_+, u)$$

is a Smith order-unit space. □

**Proposition 3.4.9.** *For all maps  $f : (X, \alpha_X) \rightarrow (Y, \alpha_Y)$  in  $\mathcal{EM}(\mathcal{D})$ ,  $\text{BAff}(f)$  is a continuous map of Smith spaces, so  $\text{BAff}$  is a functor  $\mathcal{EM}(\mathcal{D}) \rightarrow \mathbf{SOUS}^{\text{op}}$ .*

*Proof.* We show that  $\text{BAff}(f) : \text{BAff}(Y) \rightarrow \text{BAff}(X)$  is continuous from the topology  $\sigma(\text{BAff}(Y), \eta_Y(Y))$  to  $\sigma(\text{BAff}(X), \eta_X(X))$ . Therefore  $[0, 1]_{\text{BAff}(f)}$  is continuous, so by Theorem 3.4.3  $\text{BAff}(f)$  is continuous in the Smith space topologies. Then  $\text{BAff}$  is a functor  $\mathcal{EM}(\mathcal{D}) \rightarrow \mathbf{SOUS}^{\text{op}}$  by the same proof that in Theorem 2.4.16 for  $\text{BAff} : \mathcal{EM}(\mathcal{D}) \rightarrow \mathbf{BOUS}^{\text{op}}$ .



We show that if  $M_{x,\epsilon}$  (where  $x \in X$  and  $\epsilon \in \mathbb{R}_{>0}$ ) is a subbasic 0-neighbourhood in  $\text{BAff}(X)$ , then  $N_{f(x),\epsilon}$  is a subbasic 0-neighbourhood in  $\text{BAff}(Y)$  such that  $N_{f(x),\epsilon} \subseteq \text{BAff}(f)^{-1}(M_{x,\epsilon})$ , which implies continuity by the fact that  $\text{BAff}(f)^{-1}$  preserves intersections.

So let  $b \in N_{f(x),\epsilon}$ , i.e.  $|b(f(x))| < \epsilon$ . Then we want to show that  $\text{BAff}(f)(b) \in M_{x,\epsilon}$ . We have that  $\text{BAff}(f)(b)(x) = b(f(x))$ . So  $|b(f(x))| < \epsilon$  implies that  $|\text{BAff}(f)(b)(x)| < \epsilon$ , so  $\text{BAff}(f)(b) \in M_{x,\epsilon}$ .  $\square$

Recall the natural isomorphisms  $\rho : F \Rightarrow \text{BAff} \circ B$  from Theorem 2.4.18, where  $\text{BAff}$  are the functors to  $\mathbf{BOUS}^{\text{op}}$ . We have seen above how to make  $\text{BAff}$  into a functor to  $\mathbf{SOUS}^{\text{op}}$ , and we have seen in Theorem 3.4.5 how to make  $F$  into a functor  $F^\sigma : \mathbf{BBNS} \rightarrow \mathbf{SOUS}^{\text{op}}$  by giving  $F(E)$  a Smith topology.

**Proposition 3.4.10.**  $\rho$  defines a natural isomorphism  $F^\sigma \Rightarrow \text{BAff} \circ B^\mathcal{D}$  and  $F^\sigma \Rightarrow \text{BAff} \circ B^\mathcal{D}_\infty$ .

*Proof.* We show this for  $B^\mathcal{D}$ , and then it follows for  $B^\mathcal{D}_\infty$  by Lemma 2.4.14.

We only need to show that  $\rho_E$  is a homeomorphism, i.e. it is continuous and open. To do this, we show that the image of the  $\sigma(F^\sigma, F)$  topology on  $F^\sigma$  is the  $\sigma(\text{BAff}(B^\mathcal{D}(E)), \eta_{B^\mathcal{D}(E)}(B^\mathcal{D}(E)))$  topology. First observe that  $\sigma(F^\sigma, F) = \sigma(F^\sigma, B^\mathcal{D}(E))$  as  $\text{span}(B^\mathcal{D}(E)) = F$ . So the family of sets  $N_{x,\epsilon}$  where  $x \in B^\mathcal{D}(E)$  and  $\epsilon \in \mathbb{R}_{>0}$  is a subbasis for  $\sigma(F^\sigma, F)$ . We have

$$\begin{aligned} \rho_E(N_{x,\epsilon}) &= \{\rho_E(a) \mid a \in N_{x,\epsilon}\} = \{\rho_E(a) \mid a \in E^\sigma \text{ and } |a(x)| < \epsilon\} \\ &= \{\rho_E(a) \mid a \in E^\sigma \text{ and } |\rho_E(a)(x)| < \epsilon\} \\ &= \{a \in \text{BAff}(B^\mathcal{D}(E)) \mid |a(x)| < \epsilon\} \\ &= \{a \in \text{BAff}(B^\mathcal{D}(E)) \mid |\eta_{B^\mathcal{D}(E)}(x)(a)| < \epsilon\} = M_{x,\epsilon} \end{aligned}$$

which is a subbasic neighbourhood for  $\sigma(\text{BAff}(B^\mathcal{D}(E)), \eta_{B^\mathcal{D}(E)}(B^\mathcal{D}(E)))$ . In the other direction, if we start with a subbasic neighbourhood for the locally convex topology  $\sigma(\text{BAff}(B^\mathcal{D}(E)), \eta_{B^\mathcal{D}(E)}(B^\mathcal{D}(E)))$ , it is always the image of a subbasic neighbourhood for  $\sigma(E^\sigma, B^\mathcal{D}(E))$ , so the two topologies are the same. Since the unit ball of  $E^\sigma$  is mapped to the unit ball of  $\text{BAff}(B^\mathcal{D}(E))$  by  $\rho_E$ , it is a linear homeomorphism of the Smith topologies as well.

The proof of naturality then carries over from Theorem 2.4.18.  $\square$

We define the functor  $\text{CStat} : \mathbf{SOUS}^{\text{op}} \rightarrow \mathcal{EM}(\mathcal{D}_\infty)$  to be  $B^\mathcal{D}_\infty \circ G^\beta$ , we reuse the name for  $\text{CStat} : \mathbf{SOUS}^{\text{op}} \rightarrow \mathcal{EM}(\mathcal{D})$ , defined as  $B^\mathcal{D} \circ G^\beta$ , and  $\text{CStat} : \mathbf{SOUS}^{\text{op}} \rightarrow \mathbf{BConv}$ , defined as  $B \circ G^\beta$ , as it will be clear from context

which is meant. It is also clear that  $\text{CStat}(A, A_+, u)$  consists of continuous positive linear functionals  $\phi : A \rightarrow \mathbb{R}$  such that  $\phi(u) = 1$ , *i.e.* continuous states, hence the name.

We define  $\eta_X : X \rightarrow \text{CStat}(\text{BAff}(X))$  as

$$\eta_X(x)(a) = a(x),$$

where  $(X, \alpha_X)$  can be a  $\mathcal{D}$ -algebra or a  $\mathcal{D}_\infty$ -algebra,  $x \in X$  and  $a \in \text{BAff}(X)$ . We also define  $\epsilon_A : A \rightarrow \text{BAff}(\text{CStat}(A))$ .

$$\epsilon_A(a)(\phi) = \phi(a),$$

where  $A$  is a Smith order-unit space,  $a \in A$ , and  $\phi \in \text{CStat}(A)$ .

For ease of notation, from now on we use  $\mathcal{D}_?$  to refer to either  $\mathcal{D}_\infty$  or  $\mathcal{D}$  in cases where the proof works both ways.

**Theorem 3.4.11.** *BAff is a left adjoint to CStat, whether we are using  $\mathcal{EM}(\mathcal{D}_\infty)$  or  $\mathcal{EM}(\mathcal{D})$ , with  $\eta$  and  $\epsilon$  being given by the definitions above, and  $\epsilon$  is a natural isomorphism.*

*Proof.* We first show that  $\eta$  and  $\epsilon$  are natural transformations, then that they satisfy the triangle identities necessary for an adjunction (from Theorem 0.4.1 (v)).

We first need to show that  $\eta_X(x) \in \text{CStat}(\text{BAff}(X))$ . It is a state by Lemma 3.4.6. We know from Proposition 3.4.8 that the Smith topology for  $\text{BAff}(X)$  is the Smithification of  $\sigma(\text{BAff}(X), \eta_X)$  with respect to  $[-u, u]$ . By definition  $\eta_X(x)$  is  $\sigma(\text{BAff}(X), \eta_X(x))$ -continuous, so  $\eta_X(x)|_{[-u, u]}$  is continuous, so by Proposition 3.2.15  $\eta_X(x)$  is a continuous state.

We want to show that  $\eta_X$  is an  $\mathcal{EM}(\mathcal{D}_?)$ -morphism, where  $\mathcal{D}_?$  means either  $\mathcal{D}$  or  $\mathcal{D}_\infty$ , as appropriate. To show this we need to show

$$\begin{array}{ccc} \mathcal{D}_?(X) & \xrightarrow{\mathcal{D}_?(\eta_X)} & \mathcal{D}_?(\text{CStat}(\text{BAff}(X))) \\ \alpha_X \downarrow & & \downarrow \alpha_{\text{CStat}(\text{BAff}(X))} \\ X & \xrightarrow{\eta_X} & \text{CStat}(\text{BAff}(X)) \end{array}$$

commutes. So let  $\psi \in \mathcal{D}_?(X)$ , and  $a \in \text{BAff}(X)$ . For the lower left path we get

$$\eta_X(\alpha_X(\psi))(a) = a(\alpha_X(\psi)) = \sum_{x \in X} \psi(x)a(x).$$

For the upper right path we get

$$\begin{aligned} \alpha_{\text{CStat}(\text{BAff}(X))}(\mathcal{D}_?( \eta_X)(\psi))(a) &= \sum_{\phi \in \text{CStat}(\text{BAff}(X))} \mathcal{D}_?( \eta_X)(\psi)(\phi) \cdot \phi(a) \\ &= \sum_{\phi \in \text{CStat}(\text{BAff}(X))} \left( \sum_{x \in \eta_X^{-1}(\phi)} \psi(x) \right) \phi(a). \end{aligned}$$

If  $\eta_X^{-1}(\phi) = \emptyset$  then the inner sum is zero and so can be omitted, so we can restrict the outer sum to be over  $\eta_X(X)$ . We pick an  $x_\phi$  for each  $\phi \in \eta_X(X)$ , such that  $\eta_X(x_\phi) = \phi$ . Resuming,

$$\begin{aligned} &= \sum_{\phi \in \eta_X(X)} \left( \sum_{x \in \eta_X^{-1}(\phi)} \psi(x) \right) \eta_X(x_\phi)(a) \\ &= \sum_{x \in X} \psi(x) \cdot \eta_X(x_{\eta_X(x)})(a) \\ &= \sum_{x \in X} \psi(x) \eta_X(x)(a) && \text{because } \eta_X(x_{\eta_X(x)}) = \eta_X(x) \\ &= \sum_{x \in X} \psi(x) a(x). \end{aligned}$$

Therefore  $\eta_X(x)$  is an  $\mathcal{EM}(\mathcal{D}_?)$ -morphism.

The naturality diagram of  $\eta$  is

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \text{CStat}(\text{BAff}(X)) \\ f \downarrow & & \downarrow \text{CStat}(\text{BAff}(f)) \\ Y & \xrightarrow{\eta_Y} & \text{CStat}(\text{BAff}(Y)). \end{array}$$

For  $X, Y \in \mathcal{EM}(\mathcal{D}_?)$ ,  $x \in X$  and  $b \in \text{BAff}(Y)$ , then for the lower left path we have

$$\eta_Y(f(x))(b) = b(f(x)),$$

while for the upper right path, we have

$$\text{CStat}(\text{BAff}(f))(\eta_X(x))(b) = \eta_X(x)(\text{BAff}(f)(b)) = \text{BAff}(f)(b)(x) = b(f(x)),$$

so the diagram commutes, and so  $\eta$  is natural.

We now show that  $\epsilon$  is a natural isomorphism  $\text{Id} \circ \text{BAff} \circ \text{CStat}$ . To avoid confusion, we temporarily use the notation  $\epsilon'$  for the counit from Theorem 3.4.5. We have that  $\rho_{G^\beta} \circ \epsilon'$  is a natural isomorphism  $\text{Id} \Rightarrow \text{BAff} \circ B^{\mathcal{D}^?} \circ G^\beta$ , and  $\text{BAff} \circ B^{\mathcal{D}^?} \circ G^\beta = \text{BAff} \circ \text{CStat}$ . Therefore  $\rho_{G^\beta} \circ \epsilon'$  has the type that we want  $\epsilon$  to have, and it is a natural isomorphism by Theorem 3.4.5 and Proposition 3.4.10. Now, if  $A$  is a Smith order-unit space,  $a \in A$ , and  $\phi \in \text{CStat}(A)$ , we have

$$\rho_{G^\beta(A)}(\epsilon'_A(a))(\phi) = \epsilon'_A(a)(\phi) = \phi(a) = \epsilon_A(a)(\phi),$$

so  $\epsilon = \rho_{G^\beta} \circ \epsilon'$  and is therefore a natural isomorphism.

The diagrams for an adjunction  $\text{BAff} \dashv \text{CStat}$  are

$$\begin{array}{ccc} \text{CStat}(A) & \xrightarrow{\eta_{\text{CStat}(A)}} & \text{CStat}(\text{BAff}(\text{CStat}(A))) \\ & \searrow \text{id}_{\text{CStat}(A)} & \downarrow \text{CStat}(\epsilon_A) \\ & & \text{CStat}(A) \end{array}$$
  

$$\begin{array}{ccc} \text{BAff}(X) & \xleftarrow{\text{BAff}(\eta_X)} & \text{BAff}(\text{CStat}(\text{BAff}(X))) \\ & \nwarrow \text{id}_{\text{BAff}(X)} & \uparrow \epsilon_{\text{BAff}(X)} \\ & & \text{BAff}(X) \end{array}$$

where  $A$  is a Smith order-unit space, and  $X$  an Eilenberg-Moore algebra of  $\mathcal{D}^?$ .

Let  $\phi \in \text{CStat}(A)$  and  $a \in A$ . Then

$$\text{CStat}(\epsilon_A)(\eta_{\text{CStat}(A)}(\phi))(a) = \eta_{\text{CStat}(A)}(\phi)(\epsilon_A(a)) = \epsilon_A(a)(\phi) = \phi(a),$$

so  $\text{CStat}(\epsilon_A)(\eta_{\text{CStat}(A)}(\phi)) = \phi$  and the top diagram commutes. Now, if we have  $a \in \text{BAff}(X)$  and  $x \in X$ ,

$$\text{BAff}(\eta_X)(\epsilon_{\text{BAff}(X)}(a))(x) = \epsilon_{\text{BAff}(X)}(a)(\eta_X(x)) = \eta_X(x)(a) = a(x),$$

so the bottom diagram commutes as well.  $\square$

As  $B : \mathbf{BBNS} \rightarrow \mathbf{CBConv}$  is an equivalence (Proposition 2.4.13), Theorem 0.4.3 implies the existence of a functor  $\text{Emb} : \mathbf{CBConv} \rightarrow \mathbf{BBNS}$  such that there exist a unit and counit making  $B$  and  $\text{Emb}$  part of an adjoint equivalence, and we have both  $B \dashv \text{Emb}$  and  $\text{Emb} \dashv B$ .

**Corollary 3.4.12.** *The functors  $B^{\mathcal{D}_?} : \mathbf{BBNS} \rightarrow \mathcal{EM}(\mathcal{D}_?)$  and the comparison functor  $\mathbf{CBConv} \rightarrow \mathcal{EM}(\mathcal{D}_?)$  have left adjoints, making  $\mathbf{BBNS}$  and  $\mathbf{CBConv}$  full reflective subcategories of  $\mathcal{EM}(\mathcal{D}_?)$ .*

*Proof.* We have that  $\mathbf{BAff} \dashv \mathbf{CStat} = B^{\mathcal{D}_?} \circ G^\beta$ , and  $G^\beta \dashv F^\sigma$  by Theorems 3.4.11 and 3.4.5 respectively. By composing the adjunctions, we have that  $G^\beta \circ \mathbf{BAff} \dashv B^{\mathcal{D}_?} \circ G^\beta \circ F^\sigma$ . Applying Theorem 3.4.5 again, we have the isomorphism  $B^{\mathcal{D}_?} \circ G^\beta \circ F^\sigma \cong B^{\mathcal{D}_?}$ , so  $G^\beta \circ \mathbf{BAff} \dashv B^{\mathcal{D}_?}$ .

The comparison functor  $K : \mathbf{CBConv} \rightarrow \mathcal{EM}(\mathcal{D}_?)$  is  $B^{\mathcal{D}_?} \circ \mathbf{Emb}$ . We can compose the previous adjunction with  $B \dashv \mathbf{Emb}$  to get the adjunction  $B \circ G^\beta \circ \mathbf{BAff} \dashv B^{\mathcal{D}_?} \circ \mathbf{Emb}$ . Using our definitions, this is  $\mathbf{CStat} \circ \mathbf{BAff} \dashv K$ .  $\square$

We now have three sides of the square (3.2). To fill in the fourth side, we define  $\mathbf{CStat} : \mathbf{CEMod}^{\text{op}} \rightarrow \mathbf{CBConv}$  to be  $B(G^\beta(A))$  for all compact effect modules of the form  $[0, 1]_A$ . As  $[0, 1]_-$  is an equivalence, this can be extended to all of  $\mathbf{CEMod}^{\text{op}}$ , all such extensions being naturally isomorphic. This implies  $\mathbf{CStat} \circ [0, 1]_- = B \circ G^\beta$ . We also define  $\mathbf{BAff}(-, [0, 1])$  as

$$\begin{aligned} \mathbf{BAff}(X, [0, 1]) &= \{a \in \mathbf{BAff}(X) \mid \forall x \in X. 0 \leq a(x) \leq 1\} \\ \mathbf{BAff}(f, [0, 1])(b) &= b \circ f \end{aligned}$$

where  $X, Y$  are objects of  $\mathbf{CBConv}$ ,  $f : X \rightarrow Y$  is an affine map and the function  $b \in \mathbf{BAff}(Y)$ . It is clear that  $\mathbf{BAff}(-, [0, 1]) = [0, 1]_- \circ \mathbf{BAff}$ , and therefore a functor.

**Theorem 3.4.13.** *In (3.2) we have  $\mathbf{BAff}(-, [0, 1]) \circ B \cong [0, 1]_- \circ F^\sigma$  and that  $\mathbf{BAff}(-, [0, 1])$  and  $\mathbf{CStat}$  define an equivalence between  $\mathbf{CBConv}$  and  $\mathbf{CEMod}^{\text{op}}$ .*

*Proof.* We have a natural isomorphism  $\rho : F^\sigma \Rightarrow \mathbf{BAff} \circ B$  from Proposition 3.4.10. Therefore  $[0, 1]_\rho : [0, 1]_- \circ F^\sigma \Rightarrow [0, 1]_- \circ \mathbf{BAff} \circ B = \mathbf{BAff}(-, [0, 1]) \circ B$ , which is the required isomorphism.

To prove that  $\mathbf{BAff}(-, [0, 1])$  and  $\mathbf{CStat}$  define an equivalence, we show that  $\mathbf{BAff}(-, [0, 1]) \circ \mathbf{CStat} \cong \text{Id}_{\mathbf{CEMod}}$  and  $\mathbf{CStat} \circ \mathbf{BAff}(-, [0, 1]) \cong \text{Id}_{\mathbf{CBConv}}$ . We reason as follows

$$\begin{aligned} \mathbf{BAff}(-, [0, 1]) \circ B &\cong [0, 1]_- \circ F^\sigma && \Leftrightarrow \\ \mathbf{BAff}(-, [0, 1]) \circ B \circ \mathbf{Emb} &\cong [0, 1]_- \circ F^\sigma \circ \mathbf{Emb} && \Leftrightarrow \\ \mathbf{BAff}(-, [0, 1]) &\cong [0, 1]_- \circ F^\sigma \circ \mathbf{Emb}, \end{aligned}$$

as  $\text{Emb}$  is an inverse for  $B$ . Similarly, we have

$$\begin{aligned} \text{CStat} \circ [0, 1]_- &= B \circ G^\beta && \Leftrightarrow \\ \text{CStat} \circ [0, 1]_- \circ \mathcal{T} &= B \circ G^\beta \circ \mathcal{T} && \Leftrightarrow \\ \text{CStat} &\cong B \circ G^\sigma \circ \mathcal{T}, \end{aligned}$$

as  $\mathcal{T}$  is an inverse for  $[0, 1]_-$ .

Then we have

$$\begin{aligned} \text{CStat} \circ \text{BAff}(-, [0, 1]) &\cong B \circ G^\beta \circ \mathcal{T} \circ [0, 1]_- \circ F^\sigma \circ \text{Emb} \\ &\cong B \circ G^\beta \circ F^\sigma \circ \text{Emb} && \text{Theorem 3.4.3} \\ &\cong B \circ \text{Emb} && \text{Theorem 3.4.5} \\ &\cong \text{Id}_{\mathbf{CBConv}} && \text{Proposition 2.4.13.} \end{aligned}$$

On the other side, we have

$$\begin{aligned} \text{BAff}(-, [0, 1]) \circ \text{CStat} &\cong [0, 1]_- \circ F^\sigma \circ \text{Emb} \circ B \circ G^\beta \circ \mathcal{T} \\ &\cong [0, 1]_- \circ F^\sigma \circ G^\beta \circ \mathcal{T} && \text{Proposition 2.4.13} \\ &\cong [0, 1]_- \circ \mathcal{T} && \text{Theorem 3.4.5} \\ &\cong \text{Id}_{\mathbf{CEMod}} && \text{Theorem 3.4.3.} \end{aligned}$$

□

### 3.5 Universal Enveloping Objects

We can combine the adjunction<sup>4</sup>  $F \dashv G$  and the adjoint equivalences  $F^\sigma \dashv G^\beta$  and  $F^\beta \dashv G^\sigma$  to produce adjunctions analogous to the enveloping  $W^*$ -algebra of a  $C^*$ -algebra.

By Corollary 3.2.18 each Smith base-norm space  $(E, \mathcal{T}, E_+, \tau)$  has an underlying Banach base-norm space defined by  $U(E) = (E, E_+, \tau)$ , and each Smith order-unit space  $(A, \mathcal{T}, A_+, u)$  has an underlying Banach order-unit space defined by  $V(A) = (A, A_+, u)$  (in each case the positive cone remains closed because the topology is finer). As the maps in **SBNS** and **SOUS** are maps in **BBNS** and **BOUS**, required to be continuous, we have forgetful functors  $U : \mathbf{SBNS} \rightarrow \mathbf{BBNS}$  and  $V : \mathbf{SOUS} \rightarrow \mathbf{BOUS}$ . We also see that by definition  $G = UG^\sigma$  and  $F = V^{\text{op}}F^\sigma$ .

---

<sup>4</sup> $F : \mathbf{PreBNS} \rightarrow \mathbf{BOUS}^{\text{op}}$  and  $G : \mathbf{OUS}^{\text{op}} \rightarrow \mathbf{PreBNS}$

**Theorem 3.5.1.** *The functor  $U \cong GF^\beta$ , and  $V^{\text{op}} \cong FG^\beta$ . By composition of adjunctions,  $G^\sigma F \dashv U$  and  $F^\sigma G \dashv V^{\text{op}}$ , or equivalently  $V \dashv (F^\sigma G)^{\text{op}}$ .*

*Proof.* From Theorem 3.3.7, we have the isomorphism  $\eta : \text{Id}_{\mathbf{SBNS}} \Rightarrow G^\sigma F^\beta$ , so  $U\eta : U \Rightarrow GF^\beta$  is a natural isomorphism. Analogously,  $V^{\text{op}}\epsilon : V^{\text{op}} \rightarrow FG^\beta$  is a natural isomorphism, using the  $\epsilon$  from Theorem 3.4.5. If we compose the adjunctions from the above two theorems and the one from Theorem 2.5.4, we have  $G^\sigma F \dashv GF^\beta \cong U$  and  $V^{\text{op}} \cong FG^\beta \dashv F^\sigma G$ , as required.  $\square$

On the underlying Banach spaces,  $G^\sigma F$  and  $(F^\sigma G)^{\text{op}}$  are both double dualization, taking a space  $E$  to  $E^{**}$ . Therefore the above theorem for  $(F^\sigma G)^{\text{op}}$  is a version of Dauns's adjunction for the universal enveloping  $W^*$ -algebra of a  $C^*$ -algebra [22, §3].

We can also consider, for an object  $(E, X) \in \mathbf{CCL}$ ,  $U'(X) = (E, X)$  is an object of  $\mathbf{CBConv}$  because compact spaces are complete in their unique uniformity [16, II.4.1 Theorem 1]. Again, as maps in  $\mathbf{CCL}$  are just maps in  $\mathbf{CBConv}$  required to be continuous,  $U'$  is a functor  $\mathbf{CCL} \rightarrow \mathbf{CBConv}$ .

**Theorem 3.5.2.** *The functor  $\text{Stat} \circ V^{\text{op}} \circ \text{BAff}$  is a left adjoint to  $U'$ .*

*Proof.* We first show that  $U' \cong BU\text{Emb}$ , as follows. By definition the following diagram commutes

$$\begin{array}{ccc} \mathbf{SBNS} & \xrightarrow{B} & \mathbf{CCL} \\ U \downarrow & & \downarrow U' \\ \mathbf{BBNS} & \xrightarrow{B} & \mathbf{CBConv}, \end{array}$$

so  $BU\text{Emb} = U'BE\text{mb}$ . By the definition of  $\text{Emb}$ , we have  $BE\text{mb} \cong \text{Id}_{\mathbf{CCL}}$ , so composing this isomorphism with  $U'$  gives  $U' \cong U'BE\text{mb} = BU\text{Emb}$ .

We can then compose adjunctions as follows

$$\begin{array}{c} \mathbf{CCL} \\ \begin{array}{c} \uparrow \\ B \dashv \text{Emb} \\ \downarrow \end{array} \\ \mathbf{SBNS} \\ \begin{array}{c} \uparrow \\ G^\sigma F \dashv U \\ \downarrow \end{array} \\ \mathbf{BBNS} \\ \begin{array}{c} \uparrow \\ \text{Emb} \dashv B \\ \downarrow \end{array} \\ \mathbf{CBConv}, \end{array}$$

showing that  $BG^\sigma F\text{Emb} \dashv U'$ . But this is not quite the statement we want, so we rearrange the left hand side a little:

$$\begin{aligned} B \circ G^\sigma \circ F \circ \text{Emb} &= \text{Stat} \circ F \circ \text{Emb} && \text{definition of Stat} \\ &= \text{Stat} \circ V \circ F^\sigma \circ \text{Emb} \\ &\cong \text{Stat} \circ V \circ \text{BAff} && \text{Proposition 3.4.10.} \end{aligned}$$

This gives  $\text{Stat} \circ V \circ \text{BAff} \dashv U'$ , as required.  $\square$

This shows that every  $(E, X) \in \mathbf{CBConv}$  has a universal compactification. The construction of it given above is an instance of Semadeni compactification [97, Theorem 4.5].

We can also consider the case of effect modules. We can define the functor  $V' : \mathbf{CEMod} \rightarrow \mathbf{BEMod}$  by dropping the embedding in a topological vector space. The reason the codomain is  $\mathbf{BEMod}$  and not just  $\mathbf{EMod}$  is Corollary 3.2.18.

**Theorem 3.5.3.** *The functor  $(\text{BAff}(-, [0, 1]) \circ U' \circ \text{Stat})^{\text{op}}$  is a left adjoint to  $V'$ .*

*Proof.* First we show that  $V' \cong [0, 1]_- \circ V \circ \mathcal{T}$ . Observe that by definition, the diagram

$$\begin{array}{ccc} \mathbf{SOUS} & \xrightarrow{[0,1]_-} & \mathbf{CEMod} \\ V \downarrow & & \downarrow V' \\ \mathbf{BOUS} & \xrightarrow{[0,1]_-} & \mathbf{BEMod} \end{array}$$

commutes. We have that  $[0, 1]_- \circ \mathcal{T} \cong \text{Id}_{\mathbf{CEMod}}$  by Theorem 3.4.3, so  $V' \cong V' \circ [0, 1]_- \circ \mathcal{T} = [0, 1]_- \circ V \circ \mathcal{T}$ .

We can then compose adjunctions as follows

$$\begin{array}{c} \mathbf{CEMod} \\ \uparrow [0,1]_- \quad \downarrow \mathcal{T} \\ \mathbf{SOUS} \\ \uparrow (F^\sigma G)^{\text{op}} \quad \downarrow V \\ \mathbf{BOUS} \\ \uparrow \mathcal{T} \quad \downarrow [0,1]_- \\ \mathbf{CEMod}, \end{array}$$



giving us  $[0, 1]_- \circ (F^\sigma G)^{\text{op}} \circ \mathcal{T} \dashv [0, 1]_- \circ V \circ \mathcal{T} \cong V'$ . We need to adjust the left hand side a bit:

$$\begin{aligned}
& [0, 1]_- \circ (F^\sigma G)^{\text{op}} \circ \mathcal{T} \\
& \cong [0, 1]_- \circ (\text{BAff} \circ B \circ G)^{\text{op}} \circ \mathcal{T} && \text{Proposition 3.4.10} \\
& = (\text{BAff}(-, [0, 1]) \circ B \circ G)^{\text{op}} \circ \mathcal{T} && \text{definition of } \text{BAff}(-, [0, 1]) \\
& = (\text{BAff}(-, [0, 1]) \circ B \circ U \circ G^\sigma)^{\text{op}} \circ \mathcal{T} \\
& = (\text{BAff}(-, [0, 1]) \circ U' \circ B \circ G^\sigma)^{\text{op}} \circ \mathcal{T} && \text{see Theorem 3.5.2} \\
& \cong (\text{BAff}(-, [0, 1]) \circ U' \circ \text{Stat})^{\text{op}},
\end{aligned}$$

the last isomorphism arising from the definition of  $\text{Stat} : \mathbf{BEMod} \rightarrow \mathbf{CCL}$  and  $(\mathcal{T}, [0, 1]_-)$  forming an equivalence between  $\mathbf{BEMod}$  and  $\mathbf{BOUS}$ .  $\square$

### 3.6 Relationship to $C^*$ and $W^*$ -algebras

We have  $\mathbf{C}^*\mathbf{Alg}_{\text{PU}}$  is a full subcategory of  $\mathbf{BOUS}$  via the functor  $\text{SA}$  (Proposition 1.2.10), so we can produce the following state-effect triangle:

$$\begin{array}{ccc}
\mathbf{BEMod}^{\text{op}} & \xrightleftharpoons[\text{CAff}(-, [0, 1])]{\text{Stat}} & \mathbf{CCL} \\
& \nwarrow [0, 1]_- \quad \nearrow \text{Stat} & \\
& \mathbf{C}^*\mathbf{Alg}_{\text{PU}}^{\text{op}} &
\end{array}$$

The top line is an adjoint equivalence by Theorem 3.3.9. As  $\text{Stat} = \text{Stat} \circ [0, 1]_-$  by definition, and

$$\text{CAff}(-, [0, 1]) \circ \text{Stat} = \text{CAff}(-, [0, 1]) \circ B \circ G^\sigma \cong [0, 1]_- \circ F^\beta \circ G^\sigma \cong [0, 1]_-,$$

the two isomorphisms being from Theorems 3.3.9 and 3.3.7, so the triangle commutes up to isomorphism<sup>5</sup>.

The state-and-effect triangle above summarizes how, given a quantum program in  $\mathbf{C}^*\mathbf{Alg}_{\text{PU}}$ , one can consider its state transformer semantics in  $\mathbf{CCL}$ , which is also known as the “Schrödinger picture”, and also its predicate transformer semantics in  $\mathbf{BEMod}^{\text{op}}$ , also known as the “Heisenberg picture”, and these are equivalent, with no further healthiness conditions. This triangle appeared first in [44], with  $\mathcal{EM}(\mathcal{R})$  instead of  $\mathbf{CCL}$ , but we will see in the next chapter that this is equivalent.

<sup>5</sup>We implicitly use the definitions  $[0, 1]_- \circ \text{SA} = [0, 1]_-$  and  $\text{Stat} \circ \text{SA} = \text{Stat}$ .

The above implies that  $\text{Stat} : \mathbf{C^*Alg}_{\text{PU}} \rightarrow \mathbf{CCL}$  is full and faithful. As an aside, we note that Alfsen, Hanche-Olsen and Shultz have characterized the essential image of  $\text{Stat}$  [5, Corollary 8.6]. We do not give the characterization here as it involves many further definitions. We note that it is also possible to produce a characterization of the spaces of *pure* states of  $\mathbf{C^*}$ -algebras [77, §I.3.9], which is closer to what happens in Gelfand duality.

Since there are PU-maps that are not completely positive,  $\text{Stat}$  is not a full functor when restricted to  $\mathbf{C^*Alg}_{\text{CPU}}$ , the category of  $\mathbf{C^*}$ -algebras with completely positive unital maps. In fact, whether a map is completely positive or not depends on the orientation (in the sense of [5]) and cannot be defined purely from the  $\mathbf{CCL}$  structure of the state space. This can be seen by the fact that the transpose map, the archetypal positive but not completely positive map, is self-inverse, and hence an isomorphism as a PU map, and so by the above result defines an isomorphism in  $\mathbf{CCL}$  on the state space.

We now move on to  $W^*$ -algebras. As we saw in Theorem 2.3.1, the self-adjoint part of the predual  $A_*$  of a  $W^*$ -algebra  $A$  can be equipped with the structure of a base-norm space such that the dual of  $A_*$  is the self-adjoint part of  $A$ , where the base is the normal state space.

A morphism of  $W^*$ -algebras, whether it is positive, completely positive, or a  $*$ -homomorphism, is said to be *normal* if it is continuous with respect to the weak- $*$  topologies arising from the preduals. We define  $\mathbf{W^*Alg}_{\text{PU}}$  to be the category with  $W^*$ -algebras as objects and normal PU-maps as morphisms. Similarly,  $\mathbf{W^*Alg}$  has normal  $*$ -homomorphisms, and  $\mathbf{W^*Alg}_{\text{P} \leq 1}$  has normal positive subunital maps. Similar to the  $\mathbf{C^*}$ -algebraic case, we can restrict to full subcategories on commutative  $W^*$ -algebras, which we call  $\mathbf{CW^*Alg}$  and  $\mathbf{CW^*Alg}_{\text{PU}}$ .

We can define  $\text{SA}$ , extending the definition from chapter 1 to  $\mathbf{W^*Alg}_{\text{PU}}$  as follows.

**Lemma 3.6.1.** *For any  $W^*$ -algebra  $A$ , the adjoint operation  $-^* : A \rightarrow \overline{A}$  is  $\sigma(A, A_*)_b$ -continuous. Therefore  $\text{SA}(A)$  is a  $\sigma(A, A_*)_b$ -closed subspace, and therefore a Smith space. The set of positive operators is  $\sigma(A, A_*)$ -closed and therefore  $\sigma(A, A_*)_b$ -closed.*

*Proof.* We show that  $-^*$  is continuous for the weak operator topology on any von Neumann algebra  $A$  on a Hilbert space  $\mathcal{H}$ . Recall that the weak operator topology is defined to be the coarsest topology such that for all  $\psi, \phi \in \mathcal{H}$  the functional  $a \in A \mapsto \langle \psi, a\phi \rangle$  is continuous [63, Definition 5.1.1]. We show that  $-^*$  is continuous by showing it preserves limits of nets. Let  $(a_i)_{i \in I}$  be a net in  $A$ , converging with respect to the weak operator topology to  $a \in A$ , i.e.

for all  $\psi, \phi \in \mathcal{H}$ , we have  $\langle \psi, a_i \phi \rangle \rightarrow \langle \psi, a \phi \rangle$  in  $\mathbb{C}$ . Then

$$\langle \psi, a_i \phi \rangle = \langle a_i^* \psi, \phi \rangle = \overline{\langle \phi, a_i^* \psi \rangle}$$

so we have  $\overline{\langle \phi, a_i^* \psi \rangle} = \overline{\langle \phi, a^* \psi \rangle}$ . As  $\bar{\cdot} : \mathbb{C} \rightarrow \mathbb{C}$  is a homeomorphism, we therefore have, for all  $\phi, \psi \in \mathcal{H}$ :

$$\langle \phi, a_i^* \psi \rangle \rightarrow \langle \phi, a^* \psi \rangle,$$

so  $a_i^* \rightarrow a^*$  in the weak operator topology. As this holds for an arbitrary  $a \in A$ , we have  $-^*$  is continuous in the weak operator topology.

The weak operator topology agrees with the  $\sigma(A, A_*)$ -topology on the unit ball [63, Theorem 7.4.2], their Smithifications are the same,  $\sigma(A, A_*)_b$ . Because  $\sigma(A, A_*)_b$  is finer than the weak operator topology,  $-^* : A \rightarrow \overline{A}$  is continuous from  $\sigma(A, A_*)_b$  to the weak operator topology. By Corollary 3.2.16, the map  $-^* : A \rightarrow \overline{A}$  is  $\sigma(A, A_*)_b$ -continuous for any von Neumann algebra  $A$ . We then use the fact that every  $W^*$ -algebra is linearly homeomorphic to a von Neumann algebra by a normal  $*$ -homomorphism [118, Theorem III.3.5]. Using Corollary 3.2.16 again, this is true for the Smithifications of the  $\sigma(A, A_*)$  and ultraweak topologies also, so  $-^*$  is continuous in this topology for all  $W^*$ -algebras.

We therefore have  $\text{SA}(A)$  is  $\sigma(A, A_*)_b$ -closed because

$$\text{SA}(A) = (-^* - \text{id}_A)^{-1}(\{0\}),$$

the preimage of a closed set by a continuous function. It is therefore a Smith space by Lemma 3.2.1.

For the positive cone, recall that a bounded operator  $a$  on a Hilbert space  $\mathcal{H}$  is positive iff for all  $\phi \in \mathcal{H}$  we have  $\langle \phi, a \phi \rangle \geq 0$  [26, §1.6.7]. Therefore this is true for any element of a von Neumann algebra  $A$  on  $\mathcal{H}$ . We therefore have

$$A_+ = \bigcap_{\phi \in \mathcal{H}} \langle \phi, - \phi \rangle^{-1}([0, \infty)),$$

which is closed in the weak operator topology because it is an intersection of closed sets. As  $\sigma(A, A_*)$  and  $\sigma(A, A_*)_b$  are finer than the weak operator topology,  $A_+$  is closed in them as well, for any von Neumann algebra. This is therefore true for all  $W^*$ -algebras by the same argument seen above.  $\square$

**Proposition 3.6.2.** *For each  $W^*$ -algebra  $A$ ,  $(\text{SA}(A), \sigma(A, A_*)_b, A_+, 1_A)$  is a Smith order-unit space. Defining  $\text{SA}(f) = f|_{\text{SA}(A)}$  for any normal PU-map or positive subunital map  $f : A \rightarrow B$  defines functors  $\mathbf{W}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathbf{SOUS}$  and  $\mathbf{W}^*\mathbf{Alg}_{\text{P}\leq 1} \rightarrow \mathbf{SOUS}_{\leq 1}$ . These functors are full and faithful.*

*Proof.* We have already seen in Proposition 1.2.10 that  $(\text{SA}(A), A_+, 1_A)$  is a Banach order-unit space, and Lemma 3.6.1 shows that it is a Smith space with respect to  $[-1_A, 1_A]$  and  $A_+$  is closed, so  $(\text{SA}, \sigma(A, A_*)_b, A_+, 1_A)$  is a Smith order-unit space.

We have  $\text{SA}(f) = f_{\text{SA}(A)}$  defines a functor in each case by combining Proposition 1.2.10 with the fact that the composition of continuous functions is continuous.

Faithfulness follows directly from the faithfulness in Proposition 1.2.10, so we only need to show fullness. Let  $g : \text{SA}(A) \rightarrow \text{SA}(B)$  be a morphism in **SOUS**. As in Proposition 1.2.10, we define

$$f(a) = g(a_{\Re}) + ig(a_{\Im}),$$

and this is a map in  $\mathbf{C}^*\mathbf{Alg}_{\text{PU}}(A, B)$ , or  $\mathbf{C}^*\mathbf{Alg}_{\text{P}\leq 1}(A, B)$  such that its restriction  $\text{SA}(f) = g$ . Therefore we only need to show that  $f$  is continuous, from  $\sigma(A, A_*)$  to  $\sigma(B, B_*)$ . We first remark that the formulas defining  $a_{\Re}$  and  $a_{\Im}$  show that the mappings  $-\Re$  and  $-\Im$  are continuous in  $\sigma(A, A_*)_b$ , because addition and scalar multiplication are continuous in any topological vector space, and  $-^*$  is continuous by Lemma 3.6.1.

Let  $(a_j)_{j \in J}$  be a net in  $A$  converging to  $a \in A$  in  $\sigma(A, A_*)_b$ . We have

$$f(a_j) = g((a_j)_{\Re}) + ig((a_j)_{\Im})$$

By the previous paragraph,  $(a_j)_{\Re} \rightarrow a_{\Re}$ , so since  $g$  is continuous  $g((a_j)_{\Re})$  converges to  $g(a_{\Re})$ , and similarly  $g((a_j)_{\Im})$  converges to  $g(a_{\Im})$ . By continuity of addition and scalar multiplication

$$f(a_j) = g((a_j)_{\Re}) + ig((a_j)_{\Im}) \rightarrow g(a_{\Re}) + ig(a_{\Im}) = f(a),$$

and therefore  $f$  is continuous from  $\sigma(A, A_*)_b$  to  $\sigma(B, B_*)_b$ . By Corollary 3.2.24,  $f$  is continuous from  $\sigma(A, A_*)$  to  $\sigma(B, B_*)$  so is a map in  $\mathbf{W}^*\mathbf{Alg}_{\text{PU}}$ .  $\square$

**Corollary 3.6.3.** *The functor  $\text{NS} : \mathbf{W}^*\mathbf{Alg}_{\text{PU}} \rightarrow \mathcal{EM}(\mathcal{D})$  and the functor  $\text{NS}_{\leq 1} : \mathbf{W}^*\mathbf{Alg}_{\text{P}\leq 1} \rightarrow \mathcal{EM}(\mathcal{D}^{\leq 1})$  are fully faithful.*

*Proof.* The categories of  $\mathbf{W}^*$ -algebras embed fully and faithfully in **SOUS** and **SOUS** $_{\leq 1}$  by the functor  $\text{SA}$  (Proposition 3.6.2). The functor  $\text{NS} = B \circ G^{\beta} \circ \text{SA}$  and  $\text{NS}_{\leq 1} = B^{\leq 1} \circ G^{\beta} \circ \text{SA}$ . Now,  $G^{\beta}$  is an equivalence by Theorem 3.4.5, hence is full and faithful. By Proposition 2.4.8,  $B$  and  $B^{\leq 1}$  are full and faithful, so the composite functors  $\text{NS}$  and  $\text{NS}_{\leq 1}$  are full and faithful.  $\square$

The preceding corollary has been used in [102], where it is used to show that the category of  $W^*$ -algebras with *completely positive* unital maps embeds contravariantly into the category of quantum predomains defined there.

We also get a triangle of adjunctions for the state and predicate transformer semantics of quantum programs interpreted in  $\mathbf{W}^*\mathbf{Alg}_{\mathbf{PU}}$ :

$$\begin{array}{ccc}
 \mathbf{CEMod}^{\text{op}} & \begin{array}{c} \xleftarrow{\text{CStat}} \\ \xrightarrow{\text{BAff}(-, [0,1])} \end{array} & \mathbf{CBConv} \\
 & \swarrow [0,1]_- \quad \searrow \text{NS} & \\
 & \mathbf{W}^*\mathbf{Alg}_{\mathbf{PU}}^{\text{op}} &
 \end{array}$$

This time we use Theorems 3.4.13 and 3.4.5, as well as the necessary results about  $W^*$ -algebras above. We should mention that, using an alternative definition of normal map of  $W^*$ -algebras in order-theoretic terms, Mathys Renela produced a state-and-effect triangle for  $\mathbf{W}^*\mathbf{Alg}_{\mathbf{P} \leq 1}$  in [101, Theorem 4.1]. However, the top line is only an adjunction, rather than an equivalence. We do not know whether the order-theoretic definition of normal map can be used to produce an equivalence in the general case, as the proof for  $W^*$ -algebras uses the representation as operators on a Hilbert space, which is why we use the approach via Smith order-unit spaces.



## Chapter 4

# Compact Convex Sets, $\mathcal{R}$ and $\mathcal{E}$

*This chapter originated in the paper “The Expectation Monad in Quantum Foundations”[60] by Bart Jacobs, Jorik Mandemaker and the author, as well as its original version [59] with only Jacobs and Mandemaker. The part on the Radon monad originates in [44]. The part on compact effect modules is original.*

### 4.1 Introduction

In the last chapter we saw that sequentially complete bounded convex subsets of locally convex spaces, or equivalently the bases of Banach base norm spaces, can be embedded as a reflective subcategory of  $\mathcal{EM}(\mathcal{D})$  and  $\mathcal{EM}(\mathcal{D}_\infty)$ . We gave no corresponding result for compact convex subsets, or equivalently bases of Smith base-norm spaces. We have also seen that  $\mathcal{EM}(\mathcal{R})$  and  $\mathcal{EM}(\mathcal{E})$  can both be seen as categories of compact convex sets, and so should be related to **CCL** in some way. In this chapter, we use some results due to Świrszcz to prove that **CCL**  $\simeq \mathcal{EM}(\mathcal{R}) \simeq \mathcal{EM}(\mathcal{E})$ . This justifies the notion that these are a legitimate notion of convex set, as they occur in many different guises, and also that the embedding in a locally convex space can be considered as merely a property rather than a structure.

We can then apply these results to reformulate **CEMod** in two different ways in an embedding-independent fashion.

## 4.2 Świrszcz's Theorem for $\mathcal{R}$

In this section we show that the Radon monad arises from an adjunction in [116] enabling us to use Świrszcz's theorem 3 from that paper to show that the categories **CCL** and  $\mathcal{EM}(\mathcal{R})$  are equivalent. The adjunction in question has  $U: \mathbf{CCL} \rightarrow \mathbf{CHaus}$  as the right adjoint, and the details of the construction of the left adjoint are not given. In order to prove that  $\mathcal{R}$  is the monad arising from this adjunction, we need to know its unit and counit, so our next task is to define the left adjoint explicitly. Of course, any other left adjoint will be naturally isomorphic (Proposition 0.4.2).

We begin as follows. We define  $\dot{\mathcal{S}}: \mathbf{CHaus} \rightarrow \mathbf{CCL}$  as  $\dot{\mathcal{S}} = \text{Stat} \circ C$ . Hence  $\mathcal{R} = U \circ \dot{\mathcal{S}}$ . To show that  $\dot{\mathcal{S}}$  is the left adjoint to  $U$ , we use the unit and counit definition of an adjunction (Theorem 0.4.1 (iv)). We already know the unit,  $\eta_X: X \rightarrow U(\dot{\mathcal{S}}(X))$ , as we gave it when defining the unit of  $\mathcal{R}$ . To define the counit we use the notion of *barycentre*.

We can understand the intuitive notion of barycentre by thinking of a (Radon) probability measure  $\mu$  on the unit square  $[0, 1]^2$ . If we wanted to find the centre of mass of  $\mu$ , which we shall call  $b \in [0, 1]^2$ , we would take

$$b_x = \int_{[0,1]^2} x \, d\mu \quad \text{and} \quad b_y = \int_{[0,1]^2} y \, d\mu$$

for the  $x$  and  $y$  coordinates. We can see that  $x$  and  $y$  are continuous affine functions from  $[0, 1]^2 \rightarrow \mathbb{R}$ , assigning each point to its  $x$  and  $y$  coordinate respectively. Therefore we can rewrite the above as

$$\int_{[0,1]^2} x \, d\mu = x(b) \quad \text{and} \quad \int_{[0,1]^2} y \, d\mu = y(b).$$

In monadic terms, this means that both projections  $\pi_1, \pi_2: [0, 1]^2 \rightarrow [0, 1]$  are maps of Eilenberg-Moore algebras for the Radon monad, in the sense that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{R}([0, 1]^2) & \xrightarrow{\mathcal{R}(\pi_i)} & \mathcal{R}([0, 1]) \\ \beta \downarrow & & \downarrow \alpha \\ [0, 1]^2 & \xrightarrow{\pi_i} & [0, 1] \end{array}$$

We write  $\alpha$  for the algebra  $\nu \mapsto \int \text{id} \, d\nu$ , see also [54], and  $\beta$  for the product algebra structure, given by  $\mu \mapsto \langle \int \pi_1 \, d\mu, \int \pi_2 \, d\mu \rangle = \langle \int x \, d\mu, \int y \, d\mu \rangle$ .



In fact, any affine continuous  $\mathbb{R}$ -valued function  $a$  on  $[0, 1]^2$  can be expressed as  $\alpha x + \beta y + \gamma$  for  $\alpha, \beta, \gamma \in \mathbb{R}$ , and so

$$\begin{aligned} \int_{[0,1]^2} (\alpha x + \beta y + \gamma) d\mu &= \alpha \int_{[0,1]^2} x d\mu + \beta \int_{[0,1]^2} y d\mu + \gamma \int_{[0,1]^2} d\mu \\ &= \alpha x(b) + \beta y(b) + \gamma \\ &= (\alpha x + \beta y + \gamma)(b), \end{aligned}$$

or  $\int_{[0,1]^2} a d\mu = a(b)$  for all affine continuous functions  $a : [0, 1]^2 \rightarrow \mathbb{R}$ . If we use the linear functional definition of a Radon measure, we have motivated the following standard definition.

**Definition 4.2.1.** *If  $X \in \mathbf{CCL}$  and  $\phi \in \dot{S}(U(X))$ , then a point  $x \in X$  is a barycentre for  $\phi$  if for all continuous affine functions  $a$  from  $X \rightarrow \mathbb{R}$  we have that  $\phi(a) = a(x)$ .  $\square$*

To handle barycentres, and for some other purposes, will require the following important lemma and corollary, which are standard variants of the Hahn-Banach separation lemma and its corollaries. Taken together, they are an affine analogue, for objects in  $\mathbf{CCL}$ , of what Urysohn's lemma is for compact Hausdorff spaces. We collect them here for the convenience of the reader.

**Lemma 4.2.2.** *If  $E$  is a locally convex topological vector space,  $X$  a closed convex subset and  $Y$  a compact convex subset that is disjoint from  $X$ , then there exists a continuous linear functional  $\phi : E \rightarrow \mathbb{R}$  and  $\alpha \in \mathbb{R}$  such that  $\phi(X) \subseteq (\alpha, \infty)$  and  $\phi(Y) \subseteq (-\infty, \alpha)$ .  $\square$*

For a proof, see either [20, theorem IV.3.9] or [110, II.4.2 corollary 1].

**Corollary 4.2.3.** *Let  $(E, K) \in \mathbf{CCL}$ . In the following  $X, Y$  will be arbitrary closed disjoint convex subsets of  $K$ ,  $x, y$  arbitrary distinct points of  $K$ .*

- (i) *There is a  $\phi \in \mathbf{CAff}(K)$  and an  $\alpha \in \mathbb{R}$  such that  $\phi(X) \subseteq (\alpha, \infty)$  and  $\phi(Y) \subseteq (-\infty, \alpha)$ .*
- (ii) *There is a  $\phi \in \mathbf{CAff}(K)$  such that  $\phi(x) \neq \phi(y)$ .*
- (iii) *There is a  $\phi \in \mathbf{CCL}(K, [0, 1])$  and an  $\alpha \in \mathbb{R}$  such that  $\phi(X) \subseteq (\alpha, 1]$  and  $\phi(Y) \subseteq [0, \alpha)$ .*
- (iv) *There is a  $\phi \in \mathbf{CCL}(K, [0, 1])$  such that  $\phi(x) \neq \phi(y)$ .*

*Proof.*

- (i) Apply Lemma 4.2.2 to obtain  $\phi': V \rightarrow \mathbb{R}$  separating  $X$  from  $Y$ . Since  $K$  has the subspace topology,  $\phi = \phi'|_K$  is continuous, and since  $\phi'$  is linear,  $\phi$  is affine, hence  $\phi \in \text{CAff}(K)$ . We also keep the properties that  $\phi(X) \subseteq (\alpha, \infty)$  and  $\phi(Y) \subseteq (-\infty, \alpha)$ .
- (ii) This follows directly from (i), using the fact that points are compact and convex.
- (iii) We use (i) and obtain  $\phi' \in \text{CAff}(K)$  and  $\alpha' \in \mathbb{R}$ . Since the image of a compact space is compact, and a compact subset of  $\mathbb{R}$  is closed and bounded, the numbers

$$\beta_{\uparrow} = \sup \phi'(K) \qquad \beta_{\downarrow} = \inf \phi'(K)$$

exist, and  $\phi'$  can be considered as an affine continuous map  $K \rightarrow [\beta_{\downarrow}, \beta_{\uparrow}]$ . We define

$$\phi(k) = \frac{\phi(k) - \beta_{\downarrow}}{\beta_{\uparrow} - \beta_{\downarrow}}$$

if  $\beta_{\uparrow} \neq \beta_{\downarrow}$ , otherwise we define it without dividing by anything, though this can only happen if one of  $X$  or  $Y$  is empty. The image of  $\phi$  is contained in  $[0, 1]$ , and  $\phi$  is affine and continuous, being the composition of affine and continuous maps. We define

$$\alpha = \frac{\alpha' - \beta_{\downarrow}}{\beta_{\uparrow} - \beta_{\downarrow}}$$

again not doing the division if it is zero. We have that  $\phi(X) \subseteq (\alpha, \infty)$ , and since the image of  $\phi$  is contained in  $[0, 1]$ , this implies  $\phi(X) \subseteq (\alpha, 1]$ . The proof that  $\phi(Y) \subseteq [0, \alpha)$  is similar.

- (iv) This is proven using (iii), again using the fact that points are closed, convex sets.  $\square$

Using the properties proven above, we can start to define the counit of the adjunction. For  $(E, X) \in \mathbf{CCL}$ , we define  $\epsilon_X : \dot{\mathcal{S}}(U(X)) \rightarrow X$  to map a Radon measure  $\phi$  to a barycentre in  $X$ . As yet, we did not show that a barycentre exists for every measure or that it is unique if it exists. We address the second point first (the proof is standard).

**Lemma 4.2.4.** *For every  $\phi \in \dot{\mathcal{S}}(U(X))$  the barycentre is unique, i.e. any two barycentres of  $\phi$  are equal. Therefore  $\epsilon_X : \dot{\mathcal{S}}(U(X)) \rightarrow X$  mapping  $\phi$  to its barycentre is at least a partial function.*

*Proof.* Let  $(E, X)$  be an object of  $\mathbf{CCL}$ ,  $E$  being the locally convex space and  $X$  the compact convex subset. Suppose for a contradiction that  $x, x' \in X$  are barycentres of  $\phi \in \acute{S}(U)$ , and  $x \neq x'$ . By Corollary 4.2.3 (ii), there is an  $f \in \text{CAff}(X)$  such that  $f(x) \neq f(x')$ . Since  $x$  and  $x'$  are both barycentres of  $\phi$ ,

$$f(x) = \phi(f) = f(x')$$

a contradiction.  $\square$

In fact, it is well known that Radon measures on compact convex subsets of locally convex spaces always have barycentres [4, Proposition I.2.1 and I.2.2]. In [44] we used this theorem and then proved that the mapping  $\epsilon_X$  was continuous and affine in a separate step. Using the results of Chapter 3, we can in fact show this in one step. This result is also shown in [94, Proposition 1.1] and [37, Proposition 7.1], but we include it here for the convenience of the reader, and also because it makes a nice, and perhaps unexpected, application of Smith base-norm spaces.

In the following we use  $i_X$  to refer to the inclusion mapping  $\text{CAff}(X) \hookrightarrow C(U(X))$ , where  $(E, X) \in \mathbf{CCL}$ , which is a positive unital map by definition.

**Lemma 4.2.5.** *For each  $(E, X) \in \mathbf{CCL}$ , every Radon measure  $\phi \in \acute{S}(U(X))$  has a barycentre in  $X$ . The mapping  $\epsilon_X : \acute{S}(U(X)) \rightarrow X$  is an affine, continuous map, i.e. a map in  $\mathbf{CCL}$ .*

*Proof.* We first show that this is true for bases of Smith base-norm spaces, which by Proposition 3.3.3 gives us *essentially* all objects of  $\mathbf{CCL}$ . We show that this is true by showing that for any Smith base-norm space  $(E, \mathcal{T}, E_+, \tau)$  the composite map

$$\acute{S}(B(E)) \xrightarrow{\text{Stat}(i)} \text{Stat}(\text{CAff}(B(E))) \xrightarrow{\text{Stat}(\rho_E)} \text{Stat}(F^\beta(E)) \xrightarrow{B(\eta_E^{-1})} B(E),$$

where  $\rho_E$  is the restriction isomorphism from Proposition 3.3.5, and  $\eta_E$  the unit of the adjoint equivalence from Theorem 3.3.7, maps a Radon measure  $\phi$  to a barycentre of it in  $B(E)$ . It follows from this that  $\epsilon_{B(E)}$  is a total function and is a morphism in  $\mathbf{CCL}$ .

We need to show that for all  $\phi \in \acute{S}(B(E))$ , and all  $a \in \text{CAff}(B(E))$

$$\phi(a) = a(B(\eta_E^{-1}) \circ \text{Stat}(i \circ \rho_E)(\phi)).$$

We start with the right hand side:

$$\begin{aligned}
 a(B(\eta_E^{-1})(\text{Stat}(i \circ \rho_E)(\phi))) &= a(B(\eta_E^{-1})(\phi \circ i \circ \rho_E)) \\
 &= a(\eta_E^{-1}(\phi \circ i \circ \rho_E)) \\
 &= a(\eta_E^{-1}(\phi \circ \rho_E)) && i \text{ an inclusion} \\
 &= \eta_E(\eta_E^{-1}(\phi \circ \rho_E))(\rho_E^{-1}(a)) && \text{defn. of } \eta_E \text{ and } \rho_E \\
 &= (\phi \circ \rho_E)(\rho_E^{-1}(a)) \\
 &= \phi(a).
 \end{aligned}$$

We now extend this to all objects in **CCL**. We use Proposition 3.3.3, specifically the fact that every object  $(E, X) \in \mathbf{CCL}$  is isomorphic to  $(F, B(F))$  for some Smith base-norm space  $F$ . To avoid cumbersome notation, we prove instead that if  $(F, Y)$  is an object in **CCL** such that  $\varepsilon_Y : \dot{\mathcal{S}}(Y) \rightarrow Y$  is total and a map in **CCL**, and  $(E, X) \in \mathbf{CCL}$  is equipped with a **CCL**-isomorphism  $f : X \rightarrow Y$ , then

$$\dot{\mathcal{S}}(X) \xrightarrow{\dot{\mathcal{S}}(f)} \dot{\mathcal{S}}(Y) \xrightarrow{\varepsilon_Y} Y \xrightarrow{f^{-1}} X$$

maps a Radon measure  $\phi \in \dot{\mathcal{S}}(X)$  to its barycentre in  $X$ , and therefore  $\varepsilon_X$  is total and a morphism in **CCL**.

Therefore we want to show that for all  $\phi \in \dot{\mathcal{S}}(X)$  and  $a \in \text{CAff}(X)$  that  $\phi(a) = a(f^{-1}(\varepsilon_Y(\dot{\mathcal{S}}(f)(\phi))))$ . As  $a \circ f^{-1}$  is the composite of affine continuous functions, it is affine and continuous and therefore an element of  $\text{CAff}(Y)$ . By the definition of barycentre, we have

$$(a \circ f^{-1})(\varepsilon_Y(\dot{\mathcal{S}}(f)(\phi))) = \dot{\mathcal{S}}(f)(\phi)(a \circ f^{-1}) = \phi(a \circ f^{-1} \circ f) = \phi(a).$$

□

**Lemma 4.2.6.** *The family  $\{\varepsilon_X\}$  is a natural transformation  $\varepsilon : \dot{\mathcal{S}} \circ U \Rightarrow \text{Id}$ .*

*Proof.* We must show that

$$\begin{array}{ccc}
 \dot{\mathcal{S}}(U(X)) & \xrightarrow{\varepsilon_X} & X \\
 \dot{\mathcal{S}}(U(f)) \downarrow & & \downarrow f \\
 \dot{\mathcal{S}}(U(Y)) & \xrightarrow{\varepsilon_Y} & Y
 \end{array}$$

Suppose that  $\phi \in \dot{\mathcal{S}}(U(X))$  and  $\varepsilon_X(\phi) = x$ , i.e.  $x$  is the barycentre of  $\phi$ . It suffices to show that  $f(x)$  is the barycentre of  $\dot{\mathcal{S}}(U(f)(\phi))$ . Let  $h \in C(Y)$ , and we have by definition that

$$\dot{\mathcal{S}}(U(f))(\phi)(h) = \phi(h \circ f)$$

We want to show that if  $h$  is affine, then  $\dot{\mathcal{S}}(U(f))(\phi)(h) = h(f(x))$ , as this would show  $f(x)$  is the barycentre. Since  $h \circ f$  is the composite of continuous, affine functions, it is also continuous and affine, and so, using the fact that  $x$  is the barycentre of  $\phi$ , we have that  $\phi(h \circ f) = (h \circ f)(x) = h(f(x))$ , which is what we were required to prove.  $\square$

Taken together, the preceding three lemmas define the counit. We can now move on to showing that this is actually an adjunction.

**Theorem 4.2.7.** *The functor  $\dot{\mathcal{S}} : \mathbf{CHaus} \rightarrow \mathbf{CCL}$  is the left adjoint to  $U : \mathbf{CCL} \rightarrow \mathbf{CHaus}$*

*Proof.* We show that the unit-counit diagrams commute (Theorem 0.4.1 (v)).

The first diagram is:

$$\begin{array}{ccc} UY & \xrightarrow{\eta_{UY}} & U(\dot{\mathcal{S}}(U(Y))) \\ & \searrow \text{id}_{UY} & \downarrow U\varepsilon_Y \\ & & UY \end{array}$$

To show that it commutes, we must show that for all  $y \in UY$ ,  $y$  is the barycentre of  $\eta_{UY}(y)$ . Using the definition of  $\eta$ , we have that for any affine continuous function  $a : X \rightarrow \mathbb{R}$  that

$$\eta_{UY}(x)(a) = a(x)$$

because that is already true for any continuous functions  $a \in C(X)$ . Therefore  $x$  is the barycentre of  $\eta_{UY}(x)$ , and so the diagram commutes.

The second diagram we must consider is the following:

$$\begin{array}{ccc} \dot{\mathcal{S}}(X) & \xrightarrow{\dot{\mathcal{S}}(\eta_X)} & \dot{\mathcal{S}}(U(\dot{\mathcal{S}}(X))) \\ & \searrow \text{id}_{\dot{\mathcal{S}}(X)} & \downarrow \epsilon_{\dot{\mathcal{S}}(X)} \\ & & \dot{\mathcal{S}}(X) \end{array}$$

This time, we need to show that  $\phi \in \dot{\mathcal{S}}(X)$  is the barycentre of  $\dot{\mathcal{S}}(\eta_X)(\phi)$ . So consider an affine continuous function  $a : \dot{\mathcal{S}}(X) \rightarrow \mathbb{R}$ . We want to show that

$\dot{\mathcal{S}}(\eta_X)(\phi)(a) = a(\phi)$  for all  $\phi \in \dot{\mathcal{S}}(X)$ . To do this, we use Lemma 1.5.5. We show the diagram commutes on the convex combinations of extreme points, and since this is a dense subset, the diagram commutes by continuity. So let  $\{x_1, \dots, x_n\}$  be a finite subset of  $X$ , and

$$\sum_{i=1}^n \alpha_i \eta_X(x_i)$$

a finite convex combination of extreme points of  $\dot{\mathcal{S}}(X)$ . Now

$$\begin{aligned} \dot{\mathcal{S}}(\eta_X) \left( \sum_{i=1}^n \alpha_i \eta_X(x_i) \right) (a) &= \left( \sum_{i=1}^n \alpha_i \eta_X(x_i) \right) (a \circ \eta_X) \\ &= \sum_{i=1}^n \alpha_i \eta_X(x_i) (a \circ \eta_X) \\ &= \sum_{i=1}^n \alpha_i a(\eta_X(x_i)) \\ &= a \left( \sum_{i=1}^n \eta_X(x_i) \right) \end{aligned}$$

with the last step holding because  $a$  is an affine function.

As explained before, this shows  $\dot{\mathcal{S}}(\eta_X)(\phi)(a) = a(\phi)$  for all  $\phi \in \dot{\mathcal{S}}(X)$ , and hence the diagram commutes. Thus we have that  $\dot{\mathcal{S}}$  is the left adjoint to  $U$ .  $\square$

Now that we have defined the adjunction  $\dot{\mathcal{S}} \dashv U$ , we can move on to proving that  $\mathcal{R}$  is not only the same functor as the monad derived from  $\dot{\mathcal{S}} \dashv U$  but also the same as a monad. In order to do this, we require a few lemmas concerning the definition of  $\mu$  we gave at the start of Section 1.5. The map  $\mu$  was defined using the map  $\psi \mapsto \psi(a)$ . In fact, this map is simply  $\epsilon_{\text{SA}(A)}$ , using the counit from Theorem 3.3.8.

When defining  $\mu_X$  for the Radon monad, we were using  $\epsilon_{C_{\mathbb{R}}(X)}$  for a compact Hausdorff space  $X$ , since  $\text{SA}(C(X)) = C_{\mathbb{R}}(X)$ , the real-valued functions. We can see that

$$\mu_X(\Phi)(a) = \Phi(\epsilon_{C_{\mathbb{R}}(X)}(a)). \quad (4.1)$$

**Theorem 4.2.8.** *The monad  $\mathbf{CHaus} \rightarrow \mathbf{CHaus}$  given by  $\dot{\mathcal{S}} \dashv U$  is the Radon monad  $\mathcal{R}$ .*

*Proof.* We have by definition that  $\mathcal{R} = U\dot{\mathcal{S}}$  and  $\eta = \eta$ . Therefore we only need to show that  $\mu = U\varepsilon\dot{\mathcal{S}}$ . What we need to show, then, is that if  $X$  is a compact Hausdorff space and  $\Phi \in \dot{\mathcal{S}}(U(\dot{\mathcal{S}}(X)))$ , then  $\mu(\Phi)$  is the barycentre of  $\Phi$ . That is to say, for all  $a \in \text{CAff}(\dot{\mathcal{S}}(X))$ ,  $\Phi(a) = a(\mu_X(\Phi))$ . By Theorem 3.3.8, every  $a \in \text{CAff}(\dot{\mathcal{S}}(X))$  is of the form  $\epsilon_{C_{\mathbb{R}}(X)}(b)$  for some  $b \in C_{\mathbb{R}}(X)$ . Substituting this expression for  $a$ , we want to show that  $\Phi(\epsilon_{C_{\mathbb{R}}(X)}(b)) = \epsilon_{C_{\mathbb{R}}(X)}(b)(\mu_X(\Phi))$ . Starting with the right hand side and using (4.1) we get

$$\epsilon_{C_{\mathbb{R}}(X)}(b)(\mu_X(\Phi)) = \mu_X(\Phi)(b) = \Phi(\epsilon_{C_{\mathbb{R}}(X)}(b))$$

as required.  $\square$

The preceding theorem shows that Świrszcz's definitions can be translated into ours. Therefore we can appeal to the following theorem.

**Theorem 4.2.9** (Świrszcz). *The forgetful functor  $U: \mathbf{CCL} \rightarrow \mathbf{CHaus}$  is monadic, i.e.  $\mathbf{CCL} \simeq \mathcal{EM}(U \circ \dot{\mathcal{S}})$ . By Theorem 4.2.8,  $\mathbf{CCL} \simeq \mathcal{EM}(\mathcal{R})$ .  $\square$*

This comes from [116, Theorem 3]. A proof not using any monadicity theorems can be found in [112, Proposition 7.3], and another proof may be found in [68, Theorem 8.5] where the subprobabilistic case is also treated. Note that because of the weak map of monads  $\mathcal{R} \rightarrow \mathcal{G}$  (Theorem 1.6.8) it is also the case that any compact convex subset of a locally convex topological vector space becomes a  $\mathcal{G}$ -algebra when equipped with its Baire  $\sigma$ -algebra.

At this point the reader might wonder what rôle the embedding in a locally convex space plays. If  $X$  is a compact convex subset of a topological vector space,  $\mathcal{R}(X)$  is still defined, and we have a map  $\mathcal{D}(X) \rightarrow X$  by the convexity, so we have a partially defined map on the dense subset  $\tau_X(\mathcal{D}(X))$  (Lemma 1.5.5) of  $\mathcal{R}(X)$  to  $X$ . If it were always possible to extend this to all of  $\mathcal{R}(X)$ , for instance by proving it was uniformly continuous, then every compact convex subset of a topological vector space would be an  $\mathcal{R}$ -algebra, and therefore be embeddable in a locally convex space. However, there are compact convex subsets of non-locally compact spaces that have no extreme points. If they were embeddable in a locally convex space this would contradict the Krein-Milman theorem [20, Proposition 7.4]. The first example was given by Roberts [103], and later he constructed an example in  $L^p$  for  $0 < p < 1$ , a metrizable topological vector space that is not locally convex. A more recent example is given in [67, Theorem 4.3], and a textbook treatment in [105, §5.6]. Such convex sets are necessarily not “observable” in the sense of [60], i.e. there are pairs of points that cannot be distinguished by any continuous affine map to  $[0, 1]$  (or equivalently  $\mathbb{R}$ ).

### 4.3 Świrszcz's Theorem for $\mathcal{E}$

We introduce at this point the functor  $\mathbf{Set} \rightarrow \mathbf{BOUS}^{\text{op}}$  that plays, for  $\mathcal{E}$ , the rôle that  $C : \mathbf{CHaus} \rightarrow \mathbf{BOUS}^{\text{op}}$  plays for  $\mathcal{R}$ . We write  $\ell^\infty(X)$  for the set of functions  $\phi : X \rightarrow \mathbb{R}$  which are bounded: there is an  $N \in \mathbb{N}$  with  $|\phi(x)| \leq N$  for all  $x \in X$ . These functions form an ordered vector space, via pointwise operations and order. The function  $u : X \rightarrow \mathbb{R}$  with  $u(x) = 1$  is a strong unit that is archimedean. The induced norm is the *uniform* or *supremum* norm  $\|\phi\|_\infty = \sup\{|\phi(x)| \mid x \in X\}$ . It is not hard to see that  $\ell^\infty(X)$  is complete in this norm, and thus a Banach order-unit space.

For the category  $\mathbf{CHaus}$  we have seen a monadicity result over  $\mathbf{Set}$  in Subsection 0.4.1, and for  $\mathbf{CCL}$  we have seen a monadicity result over  $\mathbf{CHaus}$  (Theorem 4.2.9). There in fact a monadicity result for  $\mathbf{CCL}$  over  $\mathbf{Set}$ , also due to Świrszcz.

**Theorem 4.3.1** (Świrszcz). *The category  $\mathbf{CCL}$  of compact convex sets is monadic over  $\mathbf{Set}$ , where the left adjoint to the forgetful functor  $\mathbf{CCL} \rightarrow \mathbf{Set}$  is the following composite:*

$$\mathcal{S} = (\mathbf{Set} \xrightarrow{u} \mathbf{CHaus} \xrightarrow{\hat{S}} \mathbf{CCL}).$$

obtained by composing the adjunctions from Subsection 0.4.1 and Theorem 4.2.7.  $\square$

The proof in [116] uses Linton's monadicity theorem. A more elementary proof (of monadicity over  $\mathbf{CHaus}$ , but adaptable to this case) can be found in [112, Proposition 7.3].

However, we cannot use this result immediately because the monad arising from the adjunction above is not  $\mathcal{E}$ . To help with this, we will show that  $\text{Stat} \circ \ell^\infty \dashv V$  in the diagram below:

$$\begin{array}{ccc}
 & \mathbf{CCL} & \\
 & \uparrow \hat{S} \quad \downarrow U & \\
 \text{Stat} \circ \ell^\infty & \mathbf{CHaus} & V \\
 & \uparrow u \quad \downarrow W & \\
 & \mathbf{Set} & 
 \end{array}$$

The monad  $V\text{Stat}\ell^\infty$  will be more easily related to  $\mathcal{E}$ . We define the unit of the adjunction first.



**Lemma 4.3.2.** *The map  $\eta_X : X \rightarrow V(\text{Stat}(\ell^\infty(X)))$ , defined as follows*

$$\eta_X(x)(a) = a(x),$$

*where  $x \in X$  and  $a \in \ell^\infty(X)$ , is a natural transformation  $\eta : \text{Id} \Rightarrow V\text{Stat}\ell^\infty$ .*

*Proof.* First we must check that  $\eta_X(x)$  is a state. It is linear by the pointwise-ness of the definitions of addition and scalar multiplication. We can show it is positive because the positive elements of  $\ell^\infty(X)$  are just the non-negative functions, and the unit is 1, so  $\eta_X(x)$  preserves positive elements and the unit. Therefore  $\eta_X$  defines a function in **Set**, so we move on to verifying that it is natural. Let  $f : X \rightarrow Y$  be a function, and we want to show the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & U\text{Stat}\ell^\infty(X) \\ f \downarrow & & \downarrow U\text{Stat}\ell^\infty(f) \\ Y & \xrightarrow{\eta_Y} & U\text{Stat}\ell^\infty(Y). \end{array}$$

To do so, let  $x \in X$  and  $b \in \ell^\infty(Y)$ . For the lower left route we have

$$(\eta_Y \circ f)(x)(b) = \eta_Y(f(x))(b) = b(f(x)),$$

while for the upper right route we have

$$\begin{aligned} (U\text{Stat}\ell^\infty(f) \circ \eta_X)(x)(b) &= U(\text{Stat}(\ell^\infty(f)))(\eta_X(x))(b) = \eta_X(x)(\ell^\infty(f)(b)) \\ &= \ell^\infty(f)(b)(x) = b(f(x)), \end{aligned}$$

hence the diagram commutes.  $\square$

We can extend the notion of barycentre as follows. For  $(E, X) \in \mathbf{CCL}$  and  $\phi \in \text{Stat}(\ell^\infty(V(X)))$  we say  $x \in X$  is a *barycentre* of  $\phi$  if for all  $a \in \text{CAff}(X)$  we have  $\phi(a) = a(x)$ . As  $\text{CAff}(X) \subseteq C(X) \subseteq \ell^\infty(X)$  (Proposition A.2.3), this is a valid definition. By a similar argument to that used in Lemma 4.2.4, the Hahn-Banach separation theorem implies that barycentres are unique if they exist.

**Proposition 4.3.3.** *The functor  $\text{Stat} \circ \ell^\infty$  is left adjoint to  $V : \mathbf{CCL} \rightarrow \mathbf{Set}$ .*

*Proof.* We prove this by defining the counit and verifying the triangle axioms for the unit and counit (Theorem 0.4.1 (v)). We use, from Proposition A.2.3, the inclusion mapping  $\iota : C \Rightarrow \ell^\infty W$  (where for the moment we define the functor  $W : \mathbf{CHaus} \rightarrow \mathbf{Set}$  is the forgetful functor) to define

$$\epsilon = \varepsilon \circ \text{Stat}\iota,$$

where  $\varepsilon$  is the counit from Theorem 4.2.7. By definition this is a natural transformation of the correct type. If  $\phi \in \text{Stat}(\ell^\infty(V(X)))$  and  $a \in \text{CAff}(X)$ , we can observe that

$$\begin{aligned}\phi(a) &= \phi(\iota(a)) \\ &= \text{Stat}(\iota)(\phi)(a) \\ &= a(\varepsilon_X(\text{Stat}(\iota)(\phi))) && \text{definition of } \varepsilon \\ &= a(\epsilon_X(\phi)),\end{aligned}$$

so  $\epsilon_X$  maps states to their barycentres, where barycentre is taken with the more general sense.

Thus we have defined a unit (in Lemma 4.3.2) and a counit and need to show that they satisfy the unit-counit laws. The first diagram is the following (for  $X \in \mathbf{CCL}$ )

$$\begin{array}{ccc} VX & \xrightarrow{\eta_{VX}} & V\text{Stat}\ell^\infty VX \\ & \searrow \text{id}_{VX} & \downarrow V\epsilon_X \\ & & VX \end{array}$$

This states that the barycentre of a Dirac measure at  $x$  is  $x$ . So we must show that for all  $x \in X$ , and for all  $a \in \text{CAff}(X)$ ,  $\eta_{VX}(x)(a) = a(x)$ . This follows directly from the definition.

The second diagram is (for  $X$  a set):

$$\begin{array}{ccc} \text{Stat}(\ell^\infty(X)) & \xrightarrow{\text{Stat}(\ell^\infty(\eta_X))} & \text{Stat}\ell^\infty V\text{Stat}\ell^\infty(X) \\ & \searrow \text{id}_{\text{Stat}(\ell^\infty(X))} & \downarrow \epsilon_{\text{Stat}\ell^\infty(X)} \\ & & \text{Stat}\ell^\infty(X) \end{array}$$

Let  $\phi \in \text{Stat}(\ell^\infty(X))$ . To show this diagram commutes, we will show that  $\phi$  is the barycentre of  $\text{Stat}(\ell^\infty(\eta_X))(\phi)$ . So let  $a \in \text{CAff}(\text{Stat}(\ell^\infty(X)))$ , and we want to show that  $\text{Stat}(\ell^\infty(\eta_X))(\phi)(a) = a(\phi)$ . By Theorem 3.3.8, the affine function  $a = \epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b)$  for some  $b \in \ell^\infty(X)$ . We then have that

$$\text{Stat}(\ell^\infty(\eta_X))(\phi)(\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b)) = \phi(\ell^\infty(\eta_X)(\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b))) = \phi(\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b) \circ \eta_X).$$

By using  $x \in X$ , we observe

$$(\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b) \circ \eta_X)(x) = \epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b)(\eta_X(x)) = \eta_X(x)(b) = b(x),$$

so  $\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b) \circ \eta_X = b$ , and we have

$$\text{Stat}(\ell^\infty(\eta_X))(\phi)(\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b)) = \phi(b) = \epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(b)(\phi) = a(\phi).$$

Therefore we have an adjunction.  $\square$

Since we have just introduced this generalized notion of barycentre, we take this moment to give an equivalent characterization of it.

**Lemma 4.3.4.** *Let  $X \in \mathbf{CCL}$  and  $\phi \in \text{Stat}(\ell^\infty(V(X)))$ . A point  $x \in X$  is the barycentre of  $\phi$  iff for all  $a \in \text{CAff}(\text{Stat}(\ell^\infty(V(X))), [0, 1])$   $a(x) = \phi(a)$ . For convenience, we call this property that of being a truncated barycentre.*

*Proof.* Because  $a \in \text{CAff}(\text{Stat}(\ell^\infty(V(X))), [0, 1]) \subseteq \text{CAff}(\text{Stat}(\ell^\infty(V(X))))$ , if  $x$  is the barycentre of  $\phi$ , then for all  $a \in \text{CAff}(\text{Stat}(\ell^\infty(V(X))), [0, 1])$ , we have  $\phi(a) = a(x)$ . We reduce to proving the converse. Suppose that  $x$  is a truncated barycentre of  $\phi$ , and let  $y$  be the barycentre. Then for all functions  $a \in \text{CAff}(\text{Stat}(\ell^\infty(V(X))), [0, 1])$ , we have

$$a(x) = \phi(a) = a(y)$$

By the contrapositive of Corollary 4.2.3 (iv),  $x = y$ , so  $x$  is the barycentre of  $\phi$ .  $\square$

We now have, by Lemma 0.4.9, that  $V\text{Stat}\ell^\infty \cong V\dot{\mathcal{S}}\mathcal{U}$  as monads, so by Proposition 0.4.8 and Theorem 4.3.1  $\mathcal{EM}(V\text{Stat}\ell^\infty) \cong \mathcal{EM}(V\dot{\mathcal{S}}\mathcal{U}) \simeq \mathbf{CCL}$ . The final step is to produce a monad isomorphism  $\mathcal{E} \cong V\text{Stat}\ell^\infty$ . To do this, we first give an equivalent definition of  $\mu^\mathcal{E}$ .

We will be using the isomorphism between hom sets

$$\theta_X : \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \cong \mathbf{BEMod}([0, 1]_{\ell^\infty(X)}, [0, 1]_{\mathbb{R}})$$

that is implied to exist by Theorem 1.2.9, so we have given it a name.

**Lemma 4.3.5.** *For  $\Phi \in \mathcal{E}^2(X)$  and  $a \in [0, 1]^X$ , we have*

$$\mu^\mathcal{E}(\Phi)(a) = \Phi(W([0, 1]_{\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}}(a)) \circ \theta_X^{-1})$$

*Proof.* The original definition (1.4) is

$$\mu^\mathcal{E}(\Phi)(a) = \Phi(\phi \in \mathcal{E}(X) \mapsto \phi(a)),$$

and so it suffices to show that

$$\phi \in \mathcal{E}(X) \mapsto \phi(a) = W([0, 1]_{\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}}(a)) \circ \theta_X^{-1}$$

We start with the right hand side, evaluating it at an arbitrary  $\phi \in \mathcal{E}(X)$ :

$$\begin{aligned} (W([0, 1]_{\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}}(a)) \circ \theta_X^{-1})(\phi) &= V([0, 1]_{\epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}}(a))(\theta_X^{-1}(\phi)) \\ &= \epsilon_{\ell^\infty(X)}^{\mathbf{BOUS}}(a)(\theta_X^{-1}(\phi)) \\ &= \theta_X^{-1}(\phi)(a) = \phi(a). \end{aligned}$$

The last step is true because  $\theta_X^{-1}(\phi)$  is only being evaluated at elements of  $[0, 1]^X$ , where it agrees with  $\phi$ .  $\square$

For ease of notation, in the following we use  $(T, \eta^T, \mu^T)$  for the monad  $V\text{Stat}^{\ell^\infty}$ .

**Theorem 4.3.6.**  $\theta$  is a monad isomorphism  $T \Rightarrow \mathcal{E}$ , and so  $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCL}$ .

*Proof.* We first observe that the composite functor  $V \circ \text{Stat}$  is equal to the hom functor  $\mathbf{BOUS}(-, \mathbb{R})$ , and hence  $V\text{Stat}^{\ell^\infty} = \mathbf{BOUS}(-, \mathbb{R}) \circ \ell^\infty$ . This is simply because the set is exactly the same set of maps to  $\mathbb{R}$  in each case and the action on maps is precomposition in both cases. Now we can use  $\theta_X$  as follows:

$$\begin{aligned} V\text{Stat}^{\ell^\infty}(X) &= \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \cong \mathbf{BEMod}([0, 1]_{\ell^\infty(X)}, [0, 1]_{\mathbb{R}}) \\ &= \mathbf{BEMod}([0, 1]^X, [0, 1]) = \mathcal{E}(X) \end{aligned}$$

and this is in fact a natural isomorphism because the only actual isomorphism is  $\theta_X : \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \cong \mathbf{BEMod}([0, 1]_{\ell^\infty(X)}, [0, 1]_{\mathbb{R}})$  which was already proven to be natural, in the other cases the functors agree on both maps and objects and so are trivially naturally isomorphic.

To show it is a monad isomorphism, we must show that the relevant diagrams involving the unit and multiplication commute. First, we must show:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^T} & TX \\ & \searrow \eta_X^\mathcal{E} & \downarrow \theta_X \\ & & \mathcal{E}(X) \end{array}$$

Let  $x \in X$  and  $a \in [0, 1]^X$ . Then

$$\theta_X(\eta^T(x))(\phi) = \eta^T(x)(\phi) = \phi(x) = \eta^\mathcal{E}(x)(\phi)$$

and so the diagram commutes. We therefore move on to showing that the following diagram commutes:

$$\begin{array}{ccc}
 T^2(X) & \xrightarrow{\theta_{TX}} & \mathcal{E}(T(X)) \\
 \mu_X^T \downarrow & & \downarrow \mathcal{E}(\theta_X) \\
 T(X) & & \mathcal{E}^2(X) \\
 & \searrow \theta_X & \downarrow \mu_X^\mathcal{E} \\
 & & \mathcal{E}(X)
 \end{array}$$

The diagram commutes iff  $\mu_X^T = \theta_X^{-1} \circ \mu_X^\mathcal{E} \circ \mathcal{E}(\theta_X) \circ \theta_{TX}$ , and since, by definition,  $\mu_X^T = V\epsilon_{\text{Stat}(\ell^\infty(X))}$ , this will follow if  $\theta_X^{-1}(\mu_X^\mathcal{E}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))$  is the barycentre of  $\Phi$  for each  $\Phi \in T^2(X)$ . To simplify matters we will use the truncated barycentre characterization from Lemma 4.3.4. Suppose that  $a \in \text{CAff}(\text{Stat}(\ell^\infty(X)), [0, 1])$ , and define  $b \in [0, 1]^X$  to be such that  $[0, 1]_{\epsilon_{\ell^\infty(X)}^{\text{BOUS}}}(b) = a$  (using Theorem 3.3.8). We have

$$\begin{aligned}
 a(\theta_X^{-1}(\mu_X^\mathcal{E}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))) &= \epsilon_{\ell^\infty(X)}^{\text{BOUS}}(b)(\theta_X^{-1}(\mu_X^\mathcal{E}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))) \\
 &= \theta_X^{-1}(\mu_X^\mathcal{E}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))(b) = \mu_X^\mathcal{E}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi)))(b).
 \end{aligned}$$

We then use Lemma 4.3.5:

$$\begin{aligned}
 &= \mathcal{E}(\theta_X)(\theta_{TX}(\Phi))(W([0, 1]_{\epsilon_{\ell^\infty(X)}^{\text{BOUS}}}(b)) \circ \theta_X^{-1}) \\
 &= \theta_{TX}(\Phi)(W([0, 1]_{\epsilon_{\ell^\infty(X)}^{\text{BOUS}}}(b)) \circ \theta_X^{-1} \circ \theta_X) = \theta_{TX}(\Phi)(W([0, 1]_{\epsilon_{\ell^\infty(X)}^{\text{BOUS}}}(b))) \\
 &= \theta_{TX}(\Phi)(a) = \Phi(a).
 \end{aligned}$$

The last step works because  $\theta$  is just truncation. We can therefore deduce that  $\theta_X^{-1}(\mu_X^\mathcal{E}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))$  is the truncated barycentre of  $\Phi$ , and hence is its barycentre.

We can compose the isomorphism  $\theta^{-1} : \mathcal{E} \rightarrow V\text{Stat}^{\ell^\infty}$  with the isomorphism  $V\text{Stat}^{\ell^\infty} \rightarrow V\dot{\mathcal{S}}\mathcal{U}$  arising from Theorem 4.3.3 and Lemma 0.4.9 to obtain a monad isomorphism  $\mathcal{E} \rightarrow V\dot{\mathcal{S}}\mathcal{U}$ . By Proposition 0.4.8 we get an isomorphism of categories  $\mathcal{EM}(\mathcal{E}) \cong \mathcal{EM}(V\dot{\mathcal{S}}\mathcal{U})$ , so by Theorem 4.3.1 we have an equivalence  $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCL}$ .  $\square$

We can now show that every Eilenberg-Moore algebra of  $\mathcal{E}$  is “observable”, in the sense of [59].

**Corollary 4.3.7.** *Every  $(X, \alpha) \in \text{Obj}(\mathcal{EM}(\mathcal{E}))$  is observable, i.e. for each  $x_1, x_2 \in X$ , where  $x_1 \neq x_2$ , we have an  $a : (X, \alpha) \rightarrow [0, 1]$  in  $\mathcal{EM}(\mathcal{E})$  such that  $a(x_1) \neq a(x_2)$ .*

*Proof.* By Theorem 4.3.6, we find  $Y \in \text{Obj}(\mathbf{CCL})$  such that  $(X, \alpha) \cong (UY, \beta)$  in  $\mathcal{EM}(\mathcal{E})$ , where  $\beta$  is the Eilenberg-Moore algebra map for  $Y$ . Under the isomorphism  $X \cong U(Y)$ ,  $x_1, x_2 \in X$  map to  $y_1, y_2 \in U(Y)$  which are still distinct. By Corollary 4.2.3 (iv) there exists a  $\phi \in \mathbf{CCL}(Y, [0, 1])$  such that  $\phi(y_1) \neq \phi(y_2)$ . Regarding this as a map in  $\mathcal{EM}(\mathcal{E})$ , and composing with the isomorphism  $X \cong UY$  we have a map  $a : (X, \alpha) \rightarrow [0, 1]$  in  $\mathcal{EM}(\mathcal{E})$  such that  $a(x_1) \neq a(x_2)$ .  $\square$

## 4.4 Compact Effect Modules

Using Świrszcz's results, as shown in this chapter, we can characterize  $\mathbf{CEMod}$  intrinsically, i.e. without using an embedding in a vector space, in two different ways, according to whether we use  $\mathcal{R}$  or  $\mathcal{E}$ .

We define the objects of  $\mathbf{CEMod}_{\mathcal{R}}$  to be triples  $(A, \mathcal{T}, \alpha_A)$ , where  $A$  is an effect module,  $\mathcal{T}$  a compact Hausdorff topology on  $A$ , and  $\alpha_A : \mathcal{R}(A) \rightarrow A$  a map making  $A$  an Eilenberg-Moore algebra of  $\mathcal{R}$ , such that the  $\mathcal{EM}(\mathcal{D})$  structures on  $A$  defined by  $\alpha_A$  and the effect module structure of  $A$  are the same. The maps in  $\mathbf{CEMod}_{\mathcal{R}}$  are maps that are effect module maps that are also continuous and  $\mathcal{EM}(\mathcal{R})$  maps.

Similarly, the objects of  $\mathbf{CEMod}_{\mathcal{E}}$  are pairs  $(A, \alpha_A)$  where  $A$  is an effect module and  $\alpha_A : \mathcal{E}(A) \rightarrow A$  is a map making  $A$  an Eilenberg-Moore algebra of  $\mathcal{E}$  such that the  $\mathcal{EM}(\mathcal{D})$ -structures on  $A$  defined by  $\alpha_A$  and the effect module structure of  $A$  are the same. The maps in  $\mathbf{CEMod}_{\mathcal{E}}$  are effect module maps that are also  $\mathcal{EM}(\mathcal{E})$  maps.

We write  $U^{\mathcal{R}}$  for the comparison functor  $\mathbf{CCL} \rightarrow \mathcal{EM}(\mathcal{R})$ , known to be an equivalence by Theorem 4.2.9, and we write  $U^{\mathcal{E}}$  for the comparison functor  $\mathbf{CCL} \rightarrow \mathcal{EM}(\mathcal{E})$ , an equivalence by Theorem 4.3.6.

We can therefore define a functor  $V^{\mathcal{R}} : \mathbf{CEMod} \rightarrow \mathbf{CEMod}_{\mathcal{R}}$  and a functor  $V^{\mathcal{E}} : \mathbf{CEMod} \rightarrow \mathbf{CEMod}_{\mathcal{E}}$  as follows. The definition on objects is  $V^{\mathcal{R}}(E, A) = (A, \mathcal{T}, \alpha_A)$  where  $\mathcal{T}$  is the subspace topology on  $A$ , and  $\alpha_A$  the Eilenberg-Moore structure arising from  $U^{\mathcal{R}}(E, A)$ , considered as an object of  $\mathbf{CCL}$ . This is an object of  $\mathbf{CEMod}_{\mathcal{R}}$ , because the condition that convex combinations of elements of  $A$  defined in  $E$  agree with those defined in terms of the effect module structure implies that the  $\mathcal{EM}(\mathcal{D})$  structure defined by  $\alpha_A$  and that defined by the effect module structure agree. On maps  $V^{\mathcal{R}}$  does

nothing, and any map that is affine and continuous is an  $\mathcal{EM}(\mathcal{R})$  map, so this is well defined. The functor  $V^\mathcal{E}$  is defined similarly, based on  $U^\mathcal{E}$ .

**Proposition 4.4.1.**  *$V^\mathcal{R}$  and  $V^\mathcal{E}$  are equivalences.*

*Proof.* As  $V^\mathcal{R}$  and  $V^\mathcal{E}$  are the identity on maps, they are both faithful. We can see that each functor is full as follows. If  $g : V^\mathcal{R}(E, A) \rightarrow V^\mathcal{R}(F, B)$  is a map in  $\mathbf{CEMod}_\mathcal{R}$ , i.e. it is an effect module map  $A \rightarrow B$  that is continuous and an  $\mathcal{EM}(\mathcal{R})$  map. Then by the fullness of  $U^\mathcal{R}$ , it is an affine and continuous map  $A \rightarrow B$ , hence a morphism in  $\mathbf{CEMod}$ . In fact, we did not use the fact that it is an  $\mathcal{EM}(\mathcal{R})$ -morphism and could have defined  $\mathbf{CEMod}_\mathcal{R}$  without this condition. To see that  $V^\mathcal{E}$  is full, let  $g : V^\mathcal{E}(E, A) \rightarrow V^\mathcal{E}(F, B)$  be a map that is an effect module morphism and an  $\mathcal{EM}(\mathcal{E})$ -morphism. Then by the fullness of  $U^\mathcal{E}$ ,  $g$  is a map  $A \rightarrow B$  that is affine and continuous, considering  $A$  and  $B$  as subsets of  $E$  and  $F$  respectively, and therefore a morphism in  $\mathbf{CEMod}$ .

We now consider essential surjectivity. If  $(A, \mathcal{T}, \alpha_A) \in \mathbf{CEMod}_\mathcal{R}$ , by essential surjectivity of  $U^\mathcal{R}$ , there exists a locally convex space  $E$ , a compact convex subset  $X$ , and a map  $i : A \rightarrow X$  that is an  $\mathcal{EM}(\mathcal{R})$  isomorphism, where  $X$  is considered as an Eilenberg-Moore algebra of  $\mathcal{R}$  using the barycentre map  $\varepsilon_X : \mathcal{R}(X) \rightarrow X$ . We can define an effect module structure on  $X$  so as to make this an isomorphism, and since every  $\mathcal{EM}(\mathcal{R})$  map is  $\mathcal{D}$ -affine (Lemma 1.5.2), convex combinations defined using the effect module structure on  $X$  agree with convex combinations in  $E$ .

The proof of essential surjectivity for  $V^\mathcal{E}$  is similar, using essential surjectivity of  $U^\mathcal{E}$  and Proposition 1.5.8.  $\square$

## 4.5 Closing Remarks

In this chapter we saw two adjunctions and their composite, arranged as follows:

$$\begin{array}{c}
 \mathbf{CCL} \\
 \uparrow \quad \downarrow U \\
 \mathcal{S} \quad \dashv \\
 \mathbf{CHaus} \\
 \uparrow \quad \downarrow W \\
 \mathcal{U} \quad \dashv \\
 \mathbf{Set.}
 \end{array}$$

The monad arising from the bottom adjunction was the ultrafilter monad, the monad arising from the top adjunction was the Radon monad, and the monad

arising from the composite adjunction was the expectation monad, and all three adjunctions were monadic, giving rise to equivalences  $\mathcal{EM}(\mathcal{U}) \simeq \mathbf{CHaus}$ ,  $\mathcal{EM}(\mathcal{R}) \simeq \mathbf{CCL}$  and  $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCL}$ , the first being due to Manes and the second two to Świrszcz. Perhaps more familiar in computer science is the following pair of adjunctions and their composite, from [124]

$$\begin{array}{c}
 \mathbf{CSL} \\
 \uparrow \neg \downarrow U \\
 \mathbf{CHaus} \\
 \uparrow U \downarrow \neg W \\
 \mathbf{Set},
 \end{array}$$

where  $\mathbf{CSL}$  is the category of compact meet semilattices, or equivalently continuous lattices. The monad arising from the bottom adjunction is again the ultrafilter monad, the monad arising from the top adjunction is the Vietoris monad, and the monad arising from the composite adjunction is the *filter monad*. These adjunctions are all monadic again. This can be considered to be for nondeterminism what this chapter's results are for probability.

It might be interesting in future work to look at the relationship between these two situations and if there is a similar situation for the combination of probability and nondeterminism.



# Appendix A

## Miscellanea

*This appendix contains proofs we were unable to find elsewhere.*

### A.1 Elementary Real Analysis

**Lemma A.1.1.** *Let  $(P, \leq)$  be a directed poset, and  $(a_i)_{i \in P}$  be a monotone net in  $\mathbb{R}$ . Then*

- (i) *If  $\lim_{i \in P} a_i$  exists then  $\sup_{i \in P} a_i$  exists and  $\lim_{i \in P} a_i = \sup_{i \in P} a_i$ .*
- (ii) *If  $\sup_{i \in P} a_i$  exists then  $\lim_{i \in P} a_i$  exists and it is  $\sup_{i \in P} a_i$ .*

*Proof.*

- (i) First we show that  $\lim_{i \in P} a_i$  is an upper bound. Suppose for a contradiction that there is a  $j \in P$  such that  $a_j > \lim_{i \in P} a_i$ . We take  $\epsilon = a_j - \lim_{i \in P} a_i$ , and by our assumption this is greater than 0. Since  $(a_i)$  converges, there exists a  $k \in P$  such that for all  $m \geq j$  we have  $|a_m - \lim_{i \in P} a_i| < \epsilon$ . Since  $P$  is directed, there is some  $m \in P$  such that  $m \geq j, k$ . Since  $m \geq k$ , we have  $|a_m - \lim_{i \in P} a_i| < \epsilon$ . But since  $m \geq j$ , we have

$$a_m - \lim_{i \in P} a_i \geq a_j - \lim_{i \in P} a_i > 0,$$

so

$$|a_m - \lim_{i \in P} a_i| = a_m - \lim_{i \in P} a_i \geq a_j - \lim_{i \in P} a_i = \epsilon,$$

which is a contradiction. This implies  $\lim_{i \in P} a_i$  is an upper bound for  $(a_i)_{i \in P}$ .

To show it is a least upper bound, let  $b$  be an upper bound for  $(a_i)$ , and suppose for a contradiction that  $b < \lim_{i \in P} a_i$ . Let  $\epsilon = \lim_{i \in P} a_i - b$ . By convergence of  $(a_i)$  there is a  $j \in P$  such that  $|\lim_{i \in P} a_i - a_j| < \epsilon$ . Since  $\lim_{i \in P} a_i$  is an upper bound, we have

$$|\lim_{i \in P} a_i - a_j| = \lim_{i \in P} a_i - a_j < \epsilon = \lim_{i \in P} a_i - b.$$

Therefore  $a_j > b$ , contradicting the assumption that  $b$  is an upper bound.

- (ii) Suppose  $(a_i)$  has a least upper bound,  $\sup_{i \in P} a_i$ . We aim to show that  $(a_i)$  converges to it, i.e. that for all  $\epsilon > 0$ , there is some  $j \in P$  such that for all  $k \geq j$ ,  $|\sup_{i \in P} a_i - a_k| < \epsilon$ . So suppose for a contradiction that there is some  $\epsilon > 0$  such that for all  $j \in P$ , there exists a  $k \geq j$  such that  $|\sup_{i \in P} a_i - a_k| = \sup_{i \in P} a_i - a_k \geq \epsilon$ .

Observe that since  $a_j \leq a_k$ , we have

$$\epsilon \leq \sup_{i \in I} a_i - a_k \leq \sup_{i \in I} a_i - a_j,$$

so it is actually true that for all  $j \in P$ ,  $\sup_{i \in P} a_i - a_j \geq \epsilon > 0$ . Therefore  $\sup_{i \in P} a_i - \frac{\epsilon}{2}$  is a smaller upper bound for  $(a_i)$ , contradicting the definition of  $\sup_{i \in P} a_i$ .  $\square$

Now we can prove a lemma about monotone nets.

**Lemma A.1.2.** *Let  $(P, \leq)$  be a directed poset,  $(a_i)_{i \in P}, (b_i)_{i \in P}$  be monotone nets in  $\mathbb{R}$ , such that  $(b_i)$  converges and  $a_i \leq b_i$  for all  $i \in P$ . Then  $(a_i)$  converges.*

*Proof.* By Lemma A.1.1,  $\lim_{i \in P} b_i = \sup_{i \in P} b_i$ . Since  $a_i \leq b_i \leq \sup_{i \in P} b_i$ ,  $(a_i)$  is bounded above, and therefore has a least upper bound. Therefore  $\sup_{i \in I} a_i$  exists, so by A.1.1 again,  $\lim_{i \in P} a_i$  exists.  $\square$

## A.2 General Topology

It is commonly known that  $f : X \rightarrow Y$  is continuous iff for all nets  $(x_i)_{i \in I}$ ,  $x_i \in X$  we have  $\lim_{i \in I} f(x_i) = f(\lim_{i \in I} x_i)$ . The following lemmas relate this fact to the subspace topology.

**Lemma A.2.1.** *Let  $X$  be a topological space,  $S \subseteq X$  a subspace,  $(x_i)_{i \in I}$  a net with  $x_i \in S$  for all  $i \in I$ , and  $x \in S$ . Then  $\lim_{i \in I} x_i = x$  in  $X$ 's topology iff  $\lim_{i \in I} x_i = x$  in  $S$ 's subspace topology.*

*Proof.*

- $x_i \rightarrow x$  in  $S$  implies  $x_i \rightarrow x$  in  $X$ :

Suppose  $x_i \rightarrow x$  in  $S$ , i.e. for all  $U \in \mathcal{O}(S)$  such that  $x \in U$ , there exists  $j_U \in I$  such that for all  $i \geq j_U$  we have  $x_i \in U$ . Then if  $U \in \mathcal{O}(X)$  and  $x \in U$ ,  $U \cap S \in \mathcal{O}(S)$  so we take  $j_{U \cap S}$  to prove the convergence of  $x_i$  to  $x$ , as  $U \cap S \subseteq U$ .

- $x_i \rightarrow x$  in  $X$  implies  $x_i \rightarrow x$  in  $S$ :

Now suppose  $x_i \rightarrow x$  in  $X$ , and define  $j_U$  for each  $U \in \mathcal{O}(X)$  such that  $x \in U$  as before. If  $U \in \mathcal{O}(S)$ , we have  $U = U' \cap S$  for  $U' \in \mathcal{O}(X)$ . So if  $x \in U$ , we have  $x \in U'$  and so there exists a  $j_{U'}$  such that for all  $i \geq j_{U'}$  we have  $x_i \in U'$ . Since  $x_i \in S$ , we also have that  $x_i \in U' \cap S = U$ , so  $x_i \rightarrow x$  in  $S$  too.  $\square$

**Corollary A.2.2.** *Let  $f : X \rightarrow Y$  be a function, where  $X$  and  $Y$  are topological spaces. Let  $S \subseteq X$ . Then  $f|_S$  is continuous iff for all nets  $(x_i)_{i \in I}$  with  $x_i \in S$  that converge to a point  $x \in S$ , we have  $\lim_{i \in I} f(x_i) = f(x)$ .*

*Proof.* By [69, Chapter 3, Theorem 1] the map  $f|_S$  is continuous iff for all nets  $(x_i)_{i \in I}$  in  $S$  converging to a point  $x$  in  $S$ ,  $(f(x_i))_{i \in I}$  converges to  $f(x)$  in  $Y$ . We then use Lemma A.2.1 to deduce that this is so iff for all  $(x_i)_{i \in I}$  converging to  $x \in S$  in the topology of  $X$ ,  $(f(x_i))_{i \in I}$  converges to  $f(x)$  in the topology of  $Y$ .  $\square$

Let  $U : \mathbf{CHaus} \rightarrow \mathbf{Set}$  be the forgetful functor.

**Proposition A.2.3.** *Every continuous function on a compact Hausdorff space is bounded. Therefore we have an inclusion  $\iota_X : C(X) \rightarrow \ell^\infty(U(X))$ . This is a natural transformation  $\iota : C \Rightarrow \ell^\infty U$ . This is so whether we take  $C$  and  $\ell^\infty$  to be  $C^*$ -algebras of  $\mathbb{C}$ -valued functions or Banach order-unit spaces of  $\mathbb{R}$ -valued functions.*

*Proof.* The reason that every continuous function on a compact Hausdorff space is bounded is that the image of a compact set is compact, and every compact subset of  $\mathbb{R}$  (respectively,  $\mathbb{C}$ ) is bounded by the Heine-Borel theorem. The map  $\iota_X$  is a map in **BOUS** because the vector space structure, positive cone and order unit are exactly the same for  $C(X)$  and  $\ell^\infty(U(X))$ , or

it is a map in  $\mathbf{C^*Alg}$  because the multiplication, involution and vector space structure are the same. Thus we only need to show that this forms a natural transformation. Let  $f : X \rightarrow Y$  be a continuous map between compact Hausdorff spaces. We want to show that

$$\begin{array}{ccc} C(Y) & \xrightarrow{\iota_Y} & \ell^\infty(U(Y)) \\ C(f) \downarrow & & \downarrow \ell^\infty(U(f)) \\ C(X) & \xrightarrow{\iota_X} & \ell^\infty(U(X)) \end{array}$$

commutes.

So let  $b \in C(Y)$ , and for the lower left way:

$$\iota_X(C(f)(b)) = b \circ f,$$

while for the upper right way:

$$\ell^\infty(U(f))(\iota_Y(b)) = b \circ f$$

hence the diagram commutes.  $\square$

### A.3 Absolutely Convex Sets

**Lemma A.3.1.** *A set is absolutely convex if and only if it is balanced, convex and nonempty, and in either case contains 0.*

*Proof.* If  $E$  is a real vector space and  $S \subseteq E$  an absolutely convex set, we can see that it is convex because each convex combination is also an absolutely convex combination. It is balanced because  $-x$  is an absolutely convex combination of  $x$ . It is non-empty because we can take the empty absolutely convex combination to get  $0 \in S$ .

On the other hand, let  $S$  be balanced, convex and non-empty. We know there is an  $x \in S$ , so  $-x \in S$  by balancedness, and  $\frac{1}{2}x + \frac{1}{2}(-x) \in S$  by convexity. Therefore  $0 \in S$ .

Now, let  $\sum_{i=1}^n \alpha_i x_i$  be an absolutely convex combination of elements of  $S$ . Define  $\beta_i = |\alpha_i|$  for  $i \in \{1 \dots n\}$  and

$$\beta_{n+1} = 1 - \sum_{i=1}^n |\alpha_i|,$$

which is an element of  $[0, 1]$ . We then see that  $\sum_{i=1}^{n+1} \beta_i = 1$ . We define  $y_i = \text{sgn}(\alpha_i)x_i$  for  $i \in \{1 \dots n\}$  and  $y_{n+1} = 0$ . Then  $\sum_{i=1}^{n+1} \beta_i y_i$  is a convex combination of elements of  $S$ , and so is in  $S$  by its convexity. All we need is

$$\sum_{i=1}^{n+1} \beta_i y_i = \sum_{i=1}^n |\alpha_i| \text{sgn}(\alpha_i)x_i + \beta_{n+1}0 = \sum_{i=1}^n \alpha_i x_i$$

to show that  $S$  is absolutely convex.  $\square$

## A.4 Effect Modules

**Lemma A.4.1.** *Let  $A$  be an effect module. If  $a, b \in A$  and  $\alpha, \beta \in [0, 1]$  such that  $\alpha + \beta \leq 1$ , then  $\alpha x \perp \beta y$  in  $A$ . In particular,  $A$  is closed under convex combinations (when  $\beta = 1 - \alpha$ ). Any effect module morphism  $f : A \rightarrow B$  is an affine map with respect to these convex combinations.*

*Proof.* We have

$$\begin{aligned} \alpha u &= \alpha(a \otimes a^\perp) = \alpha a \otimes \alpha a^\perp \\ \beta u &= \beta(b \otimes b^\perp) = \beta b \otimes \beta b^\perp. \end{aligned}$$

Define  $\gamma = \alpha + \beta \in [0, 1]$ . Since  $\alpha u \otimes \beta u = \gamma u$ , we have  $\alpha a \otimes \alpha a^\perp \perp \beta b \otimes \beta b^\perp$ , and their sum is  $\gamma u$ . Using associativity, we see that  $((\alpha a \otimes \alpha a^\perp) \otimes \beta b) \perp \beta b^\perp$ . We then use commutativity and associativity as follows

$$\begin{aligned} \alpha a \otimes \alpha a^\perp \perp \beta b &\Leftrightarrow \alpha a^\perp \otimes \alpha a \perp \beta b \\ &\Rightarrow \alpha a \perp \beta b. \end{aligned}$$

To see that an effect module morphism  $f : A \rightarrow B$  is affine, observe that

$$f(\alpha x \otimes (1 - \alpha)y) = \alpha f(x) \otimes (1 - \alpha)f(y)$$

by the definition of an effect module morphism.  $\square$

**Lemma A.4.2.** *In any effect module, if  $\alpha \in [0, 1]$ , we have  $\alpha \cdot 0 = 0$ .*

*Proof.* We have that  $\alpha \cdot 0 \otimes (1 - \alpha) \cdot 0 = 0$  by the effect module axioms. Therefore  $\alpha \cdot 0 \otimes (1 - \alpha) \cdot 0 \perp 1$ . By associativity and commutativity we have therefore that  $\alpha \cdot 0 \perp 1$ , and so  $\alpha \cdot 0 = 0$  by the effect algebra axioms.  $\square$

## A.5 Order-Unit Spaces

Here we show that the two possible definitions of strong order-unit coincide.

**Lemma A.5.1.** *For an ordered vector space  $(E, E_+)$  with chosen element  $u \in E_+$ , the following two statements are equivalent:*

- (i)  $E = \bigcup_{n \in \mathbb{N}} [-nu, nu]$
- (ii)  $E_+ = \bigcup_{n \in \mathbb{N}} [0, nu]$  and  $E_+$  is generating (or  $E$  is directed).

*Proof.*

- (i)  $\Rightarrow$  (ii):

First we show that  $E_+$  is generating. Given  $x \in E$ , we have the existence of some  $n \in \mathbb{N}$  such that  $x \in [-nu, nu]$ . Therefore  $nu - x \in E_+$ . Since  $nu \in E_+$  too, we can define  $x_+ = nu$  and  $x_- = nu - x$ , and then  $x = x_+ - x_-$ .

Now we show the condition on positive elements. First,  $[0, \nu] \subseteq E_+$ , so  $\bigcup_{n \in \mathbb{N}} [0, nu] \subseteq E_+$ . For the other inclusion, suppose that  $x \in E_+$ . We have that  $x \in [-nu, nu]$  for some  $n$  by assumption, and then we simply apply the fact that  $x \geq 0$  to deduce that  $x \in [0, nu]$ .

- (ii)  $\Rightarrow$  (i):

Let  $x \in E$ . Since  $E_+$  is generating,  $x = x_+ - x_-$  for  $x_+, x_- \in E_+$ . Since this equation can be rearranged as  $x_+ - x = x_-$ , showing  $x_+ \geq x$ , and  $x - (-x_-) = x + x_- = x_+$ , showing  $x_- \leq x$ . Using the alternative definition of an order unit, there are  $m, n \in \mathbb{N}$  such that  $x_+ \in [0, mu]$  and  $x_- \in [0, nu]$ . We have that  $-x_- \in [-nu, 0]$ . We can then apply transitivity of the order and deduce  $x \in [-nu, mu]$ . If we take  $p$  to be whichever of  $\{n, m\}$  is larger, we have  $x \in [-pu, pu]$ , as required.  $\square$

Recall that  $(\{0\}, \{0\}, 0)$  is an order-unit space by our definition.

This is the reason for the apparent contradiction between the following Proposition 1.2.8 and [6, Lemma 1.15].

**Lemma A.5.2.** *If  $(A, A_+, u)$  is an order-unit space, the following three properties are equivalent.*

- (i)  $A \neq 0$
- (ii)  $u \neq 0$

(iii)  $\|u\| = 1$

*Proof.*

- (i)  $\Rightarrow$  (ii): Suppose that  $A \neq 0$ . Then there exists an  $a \in A$ , and  $a = a_+ - a_-$ , so there is some positive element  $0 \neq a' \in F$ . If  $u$  were 0, we would have  $\alpha u = 0$  for all  $\alpha \in [0, \infty)$ , contradicting its being a strong unit. Therefore  $u \neq 0$ .
- (ii)  $\Rightarrow$  (iii):  
By definition,  $u \in [-u, u]$ , so  $\|u\| \leq 1$ . Suppose that  $u \in [-\alpha u, \alpha u]$  for  $0 \leq \alpha < 1$ . Then  $u \leq \alpha u$ , i.e.  $(\alpha - 1)u \in A_+$ . Since  $u$  is positive and  $\alpha - 1 < 0$ , we have that  $-u \in A_+$  and so  $u = 0$ . By contraposition, if  $u \neq 0$ ,  $\|u\| \geq 1$ , and so  $\|u\| = 1$ .
- (iii)  $\Rightarrow$  (i):  
If  $A = 0$ , the only element is 0, and  $\|0\| = 0$ . So the existence of any element of nonzero norm implies  $A \neq 0$ .  $\square$

There is another characterization of when an ordered vector space with strong unit is an order-unit space, i.e. when it is archimedean.

**Lemma A.5.3.** *Let  $(A, A_+, u)$  be an ordered vector space with strong order unit  $u$ . Let  $\|-\| = \|-\|_{[-u, u]}$  be the Minkowski seminorm. Then  $A_+$  is  $\|-\|$ -closed iff  $(A, A_+, u)$  is archimedean. Furthermore, if  $(A, A_+, u)$  is archimedean, the seminorm  $\|-\|_{[-u, u]}$  is a norm with unit ball  $[-u, u]$ .*

*Proof.* We first show that if  $(A, A_+, u)$  is archimedean, then  $A_+$  is  $\|-\|$ -closed. Let  $x \in \text{cl}(A_+)$ . This means that for all  $n \in \mathbb{N}$ , we have that there exists a  $y \in A_+$  such that  $\|x - y\| < \frac{1}{n}$ . We know that  $\text{Ball}(A) \subseteq [-2u, 2u]$  (Lemma 0.1.6), so for all  $n \in \mathbb{N}$  we have  $x - y \in \frac{1}{n}[-2u, 2u]$ . By redefining  $n$  to  $2n$  and using Lemma 0.2.2 we have that for all  $n \in \mathbb{N}$  there exists a  $y \in A_+$  such that  $y \in [x - \frac{1}{n}u, x + \frac{1}{n}u]$ . This implies that  $x + \frac{1}{n}u - y \in A_+$  and  $y \in A_+$ , so  $x + \frac{1}{n}u \in A_+$ . Therefore  $-x \leq \frac{1}{n}u$  for all  $n \in \mathbb{N}$ , so by archimedeaness we have  $-x \in -A_+$ , so  $x \in A_+$ . Therefore  $\text{cl}(A_+) = A_+$ , i.e.  $A_+$  is closed.

We now show that if the cone is closed,  $(A, A_+, u)$  is archimedean. So suppose that for all  $n \in \mathbb{N}$ , we have  $a \leq \frac{1}{n}u$ , equivalently  $\frac{1}{n}u - a \in A_+$ . If we show that  $(\frac{1}{n}u - a)_n \rightarrow -a$ , we can conclude that  $-a \in A_+$  because  $A_+$  is closed, and therefore  $a \in -A_+$ . So let  $\epsilon \in \mathbb{R}_{>0}$ , and take  $n = \lceil \epsilon^{-1} + 1 \rceil$ . We then aim to show that for all  $i \geq n$  we have  $\|(\frac{1}{i}u - a) - (-a)\| < \epsilon$ , or equivalently  $\|\frac{1}{i}u\| < \epsilon$ . Since  $\frac{1}{i}u \in \frac{1}{i}[-u, u]$ , we have  $\|\frac{1}{i}u\| \leq \frac{1}{i}$ .

Now, since  $i \geq n$ , we have  $\frac{1}{i} \leq \frac{1}{n}$ . So we have

$$\left\| \frac{1}{i} u \right\| \leq \frac{1}{n} = \frac{1}{\lceil \epsilon^{-1} + 1 \rceil}.$$

We have

$$n = \lceil \epsilon^{-1} + 1 \rceil \geq \epsilon^{-1} + 1 > \epsilon^{-1},$$

so

$$\left\| \frac{1}{i} u \right\| \leq \frac{1}{n} < \epsilon.$$

As explained earlier, the closedness of  $A_+$  implies  $-a \in A_+$  and so  $a \in -A_+$ .

Finally, we want to show that  $\|\cdot\|_{[-u, u]}$  is a norm, and that the ball  $\text{Ball}(\|\cdot\|_{[-u, u]}) = [-u, u]$  if  $(A, A_+, u)$  is archimedean. We show  $\|\cdot\|_{[-u, u]}$  is a norm by showing  $[-u, u]$  contains no line through the origin and using Lemma 0.1.5. Suppose for a contradiction that  $x \in [-u, u]$  generates a line through the origin contained in  $[-u, u]$ . We therefore have  $nx \leq u$  for all  $n \in \mathbb{N}_{>0}$ , so  $x \leq \frac{1}{n}u$  too, and so  $x \in -A_+$ . However, this same argument can be applied with  $-nx \leq u$  to show  $-x \in -A_+$ , and since  $A_+$  is assume to be a cone,  $x = 0$ . This contradicts  $x$  generating a line.

To show  $[-u, u]$  is the unit ball, by Lemma 0.1.6,  $[-u, u] \subseteq \text{Ball}(\|\cdot\|_{[-u, u]})$ , so we show the opposite inclusion. By Lemmas 0.1.6 and 0.2.2

$$\text{Ball}(E) = \bigcap_{1 < \alpha < \infty} \alpha[-u, u] = \bigcap_{1 < \alpha < \infty} [-\alpha u, \alpha u].$$

If  $x \in \text{Ball}(E)$ , we therefore have  $x - u \leq \frac{1}{n}u$  for all  $n \in \mathbb{N}_{>0}$ , so  $x - u \in -E_+$ , i.e.  $x \leq u$ . By the same argument applied to  $-x$  we get  $-u \leq x$ , so we have  $x \in [-u, u]$ , as required.  $\square$

We will require this fact about positive linear functionals on order-unit spaces more than once.

**Lemma A.5.4.** *Let  $(A, A_+, u)$  be an order-unit space and  $\phi : A \rightarrow \mathbb{R}$  a positive linear functional. If  $\phi(u) = 0$  then  $\phi = 0$ .*

*Proof.* By linearity,  $\phi(\alpha u) = \alpha \cdot 0 = 0$  for all  $\alpha \in \mathbb{R}$ . For each  $a \in A$ , there exists  $\alpha \in \mathbb{R}_{>0}$  such that  $-\alpha u \leq a \leq \alpha u$ . As positive linear maps are monotone, we have

$$0 = \phi(-\alpha u) \leq \phi(a) \leq \phi(\alpha u) = 0,$$

$\phi(a) = 0$  for all  $a \in A$ , i.e.  $\phi = 0$ .  $\square$



## A.6 Asimow's Example

Here we give Asimow's example of a Banach base-norm space whose unit ball is not radially compact. This example was originally published in [35], without proof.

Recall the Banach space  $c_0$  of sequences of real numbers converging to 0 [28, IV.2 Example 7][20, p. 65], with pointwise vector space operations and its norm being the usual supremum norm. Recall also that there is a bilinear pairing between  $c_0$  and  $\ell^1$  defined by

$$\langle a, b \rangle = \sum_{i=0}^{\infty} a_i b_i,$$

where  $(a_i) \in c_0$  and  $(b_i) \in \ell^1$ , which defines an isomorphism  $\ell^1 \cong c_0^*$ [20, Example III.5.8].

The underlying space of Asimow's example is

$$E = \left\{ (x_i) \in c_0 \mid x_1 + x_2 = \sum_{i=3}^{\infty} \frac{x_i}{2^{i-2}} \right\}.$$

To avoid confusion, we stress at this point that the sequences are considered to start at  $x_0$ , not  $x_1$ . If we define  $\phi = (0, -1, -1, \frac{1}{2}, \dots, \frac{1}{2^{i-2}}, \dots)$ , we can see that  $\phi \in \ell^1$ , the sum of the absolute values of its entries being 3, and so  $\langle -, \phi \rangle : c_0 \rightarrow \mathbb{R}$  is a continuous linear functional. As  $E = \langle -, \phi \rangle^{-1}(0)$ , we have shown that  $E$  is a closed subset of  $c_0$ , and therefore a Banach space in the usual norm of  $c_0$ .

Following Asimow, we define  $K \subseteq E$  as

$$K = \{(x_i) \in E \mid x_0 = 1 = \|(x_i)\|_{c_0} \text{ and } \forall i \in \mathbb{N}. x_i \geq 0\},$$

and we define  $E_+$  to be the wedge generated by  $K$ , *i.e.*

$$E_+ = \{\alpha x \mid \alpha \in \mathbb{R}_{\geq 0} \text{ and } x \in K\}.$$

Take  $F = E_+ - E_+$ , or equivalently to be the span of  $K$ . We can define  $\tau : E \rightarrow \mathbb{R}$  as

$$\tau((x_i)) = x_0$$

**Proposition A.6.1.** *The subspace  $F = E$ , and  $(E, E_+, \tau)$  is a Banach base-norm space with base  $K$ .*

*Proof.* First we describe all the steps in the proof. We first show that  $K$  is convex and  $E_+$  is a proper cone. Then we show that  $\tau : F \rightarrow \mathbb{R}$  is positive and not zero, and that  $E_+ \cap \tau^{-1}(1) = K$ . Then we show  $\text{absco}(K) \subseteq \text{Ball}(\|\cdot\|_{c_0})$ , which shows that  $\text{absco}(K)$  is radially bounded and that  $F$  is a pre-base-norm space. We then show that  $\frac{1}{6}\text{Ball}(\|\cdot\|_{c_0}) \subseteq \text{absco}(K)$ , where  $\text{Ball}(\|\cdot\|_{c_0})$  is the unit ball of  $\|\cdot\|_{c_0}$  restricted to  $E$ , and therefore  $E = F$  and so  $(E, E_+, \tau)$  is a Banach pre-base-norm space. We then show that  $K$  is complete in the  $c_0$ -norm, and so therefore is  $E_+$  by Lemma 2.2.14, so  $E_+$  is closed and  $(E, E_+, \tau)$  is a Banach base-norm space.

- $K$  is convex:

Let  $(x_i), (y_i) \in K$  and  $\alpha \in [0, 1]$ . Then  $\alpha x_0 + (1 - \alpha)y_0 = \alpha + (1 - \alpha) = 1$ . Since  $x_i$  and  $y_i \geq 0$ , we have  $\alpha x_i + (1 - \alpha)y_i \geq 0$ . Then using the subadditivity of the norm

$$\|\alpha(x_i) + (1 - \alpha)(y_i)\|_{c_0} \leq \alpha\|(x_i)\| + (1 - \alpha)\|(y_i)\| = \alpha + 1 - \alpha = 1,$$

and because  $\alpha x_0 + (1 - \alpha)y_0 = 1$ , we have  $\|\alpha(x_i) + (1 - \alpha)(y_i)\| \geq 1$ , so we have shown  $\alpha(x_i) + (1 - \alpha)(y_i) \in K$ .

- $E_+$  is a proper cone:

We see immediately that  $E_+$  is closed under positive scalar multiplication. If  $\alpha x, \beta y \in E_+$  (i.e.  $x, y \in K$ ,  $\alpha, \beta \in \mathbb{R}_{\geq 0}$ ), then

$$\alpha x + \beta y = (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \right).$$

We have  $\alpha + \beta \in \mathbb{R}_{\geq 0}$ , and the rest is a convex combination of elements of  $K$ , so is an element of  $K$ . We have shown that  $E_+$  is a wedge. Now, suppose that  $\alpha(x_i) = -\beta(y_i)$ . Then in particular,  $\alpha x_0 = -\beta x_0$  and so  $\alpha = -\beta$ . As both of these are in  $\mathbb{R}_{\geq 0}$ , we have  $\alpha = \beta = 0$  and so  $E_+ \cap -E_+ = \{0\}$ , and therefore  $E_+$  is a cone.

- $\tau$  is linear, positive and nonzero:

The map  $\tau$  is linear because the vector space operations on  $E$  are point-wise. If  $\alpha(x_i) \in E_+$ , then  $\tau(\alpha(x_i)) = \alpha x_0 = \alpha \geq 0$ , so  $\tau$  is positive. If we take any element of  $(x_i) \in K$ , such as  $(x_i) = (1, 0, \dots)$ , then  $\tau((x_i)) = 1$ , so  $\tau$  is nonzero.

- $E_+ \cap \tau^{-1}(1) = K$ :

We already saw that if  $x \in K$ , we have  $\tau(x) = 1$ , and by definition  $x \in E_+$ , so we have  $K \subseteq E_+ \cap \tau^{-1}(1)$ . For the other inclusion, let  $\alpha(x_i) \in E_+$ . Then  $\tau(\alpha(x_i)) = \alpha x_0 = \alpha$ , so if  $\tau(\alpha(x_i)) = 1$ , we have  $\alpha(x_i) = (x_i)$  and therefore  $\alpha(x_i) \in K$ .

- $\text{absco}(K) \subseteq \text{Ball}(\|\cdot\|_{c_0})$  and  $(F, E_+, \tau) \in \mathbf{PreBNS}$ :

By definition of  $K$ , we have  $K \subseteq \text{Ball}(\|\cdot\|_{c_0})$ , and since  $\text{Ball}(\|\cdot\|_{c_0})$  is absolutely convex,  $\text{absco}(K) \subseteq \text{Ball}(\|\cdot\|_{c_0})$ . By Lemma 0.1.5,  $\text{Ball}(\|\cdot\|_{c_0})$  is radially bounded, so  $\text{absco}(K)$  is radially bounded. We have therefore shown  $(F, E_+, \tau)$  is a pre-base-norm space.

- $\frac{1}{6}\text{Ball}(\|\cdot\|_{c_0}) \subseteq \text{absco}(K)$ :

Recall that  $\text{Ball}(\|\cdot\|_{c_0})$  is taken to mean the unit ball of the  $c_0$ -norm restricted to  $E$ . Let  $(x_i) \in \frac{1}{6}\text{Ball}(\|\cdot\|_{c_0})$ , i.e.  $(x_i) \in E$  and  $\|x_i\| \leq \frac{1}{6}$ . It follows that for all  $i \in \mathbb{N}$ , we have  $-\frac{1}{6} \leq x_i \leq \frac{1}{6}$ . We want to produce  $(y_i), (z_i) \in K$  and  $\alpha \in [0, 1]$  such that for all  $i \in \mathbb{N}$ ,  $\alpha y_i - (1 - \alpha)z_i = x_i$ . We know that no matter what, we must take  $y_0 = z_0 = 1$ , and therefore

$$\alpha = \frac{x_0 + 1}{2},$$

and therefore the bounds on  $x_0$  give

$$\frac{5}{12} \leq \alpha \leq \frac{7}{12}$$

How we define  $y_i$  and  $z_i$  for the other coordinates depends on a case split.

- If  $x_2 \geq 0$ :

For reasons that will become clear later, we will want to define  $y_i$  so that

$$\sum_{i=3}^{\infty} 2^{-(i-2)} y_i \geq \frac{x_2}{\alpha},$$

so we want to make  $y_i$  as large as possible, while still remaining within  $c_0$ . To do this, we first define sequences  $(v_i)_{i \geq 3}, (\zeta_i)_{i \geq 3}$ , taking values in  $[0, 1]$ , such that  $\alpha v_i - (1 - \alpha)\zeta_i = x_i$  and the sum  $\sum_{i=3}^{\infty} 2^{-(i-2)} v_i > \frac{x_2}{\alpha}$ . We will ultimately define  $y_i$  and  $z_i$  using a truncation of these sequences. We define them as follows:

$$v_i = 1 \wedge \left( \frac{1 - \alpha + x_i}{\alpha} \right) \quad \zeta_i = 1 \wedge \left( \frac{\alpha - x_i}{1 - \alpha} \right),$$

where  $\wedge$  is the usual lattice operation on  $\mathbb{R}$ , *i.e.* the greatest lower bound, which is the minimum in this case.

\*  $v_i, \zeta_i \in [0, 1]$ : Since  $1 \wedge x \leq 1$  for all  $x$ , we have  $v_i, \zeta_i \leq 1$ . To show nonnegativity, we only need to show that  $\frac{1-\alpha+x_i}{\alpha} \geq 0$  and  $\frac{\alpha-x_i}{1-\alpha} \geq 0$ , because we already know  $1 \geq 0$ . As we have  $1-\alpha \geq \frac{5}{12}$  and  $x_i \geq -\frac{1}{6}$ , so  $1-\alpha+x_i \geq \frac{3}{12} \geq 0$ . Therefore  $\frac{1-\alpha+x_i}{\alpha} \geq 0$ . We have the same inequalities for  $\alpha$  and  $-x_i$ , so we have  $\frac{\alpha-x_i}{1-\alpha} \geq 0$  also.

\*  $\alpha v_i - (1-\alpha)\zeta_i = x_i$ :  
If  $\frac{1-\alpha+x_i}{\alpha} \leq 1$ , then  $\frac{\alpha-x_i}{1-\alpha} \geq 1$  and vice-versa, so there are two cases. The first is that  $v_i = \frac{1-\alpha+x_i}{\alpha}$  and  $\zeta_i = 1$ , so

$$\alpha \frac{1-\alpha+x_i}{\alpha} - (1-\alpha) \cdot 1 = x_i.$$

The other case is that  $v_i = 1$  and  $\zeta_i = \frac{\alpha-x_i}{1-\alpha}$ . Then

$$\alpha \cdot 1 - (1-\alpha) \frac{\alpha-x_i}{1-\alpha} = x_i,$$

so in either case, we are done.

\*  $\sum_{i=3}^{\infty} 2^{-(i-2)} v_i > \frac{x_2}{\alpha}$ :  
We want to show

$$\sum_{i=3}^{\infty} 2^{-(i-2)} v_i = \sum_{i=3}^{\infty} 2^{-(i-2)} \left( 1 \wedge \left( \frac{1-\alpha+x_i}{\alpha} \right) \right) > \frac{x_2}{\alpha},$$

and as  $\alpha \geq 0$ , this is equivalent to

$$\sum_{i=3}^{\infty} 2^{-(i-2)} (\alpha \wedge (1-\alpha+x_i)) > x_2.$$

We start the argument as follows. We have  $\alpha \geq \frac{5}{12}$ , and also  $1-\alpha+x_i \geq \frac{5}{12} + -\frac{1}{6} = \frac{3}{12}$ , so  $\alpha \wedge (1-\alpha+x_i) \geq \frac{3}{12}$ . Therefore

$$\sum_{i=3}^{\infty} 2^{-(i-2)} (\alpha \wedge (1-\alpha+x_i)) \geq \sum_{i=3}^{\infty} 2^{-(i-2)} \frac{3}{12} = \frac{3}{12} > \frac{1}{6} \geq x_2.$$

Convergence of the sum implies there exists an  $N \in \mathbb{N}$  such that

$\sum_{i=3}^N 2^{-(i-2)} v_i > \frac{x_2}{\alpha}$ . We define  $y_i$  and  $z_i$  as follows:

$$y_i = \begin{cases} v_i & \text{if } i \leq N \\ \frac{x_i}{\alpha} & \text{if } i > N \text{ and } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$z_i = \begin{cases} \zeta_i & \text{if } i \leq N \\ -\frac{x_i}{1-\alpha} & \text{if } i > N \text{ and } x_i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

By combining the inequalities on  $x_i$  and  $\alpha$ , we have  $\frac{x_i}{\alpha} \leq \frac{2}{5}$  and the  $x_i \geq 0$  in the case that  $\frac{x_i}{\alpha}$  occurs ensures that  $0 \leq y_i \leq 1$ . A similar argument using the inequalities for  $1-\alpha$  ensures  $0 \leq z_i \leq 1$ . We therefore have that  $y_i, z_i$  are in  $[0, 1]$  and that the sequences converge to 0 because they are eventually equal to a subsequence of a multiple of  $x_i$ , which converges to 0. We can now define

$$y_1 = \sum_{i=3}^{\infty} y_i - \frac{x_2}{\alpha} \qquad y_2 = \frac{x_2}{\alpha}$$

$$z_1 = \sum_{i=3}^{\infty} z_i \qquad z_2 = 0.$$

We first show that  $(y_i), (z_i) \in K$ . The condition that  $y_0, z_0 = 1$  is satisfied by definition. We have already shown that for  $i \geq 3$  we have  $y_i, z_i \in [0, 1]$  and  $y_i$  and  $z_i$  converge to 0.

We arranged that  $y_1 \geq 0$  by taking  $N$  to be sufficiently large so that  $\sum_{i=3}^N 2^{-(i-2)} v_i > \frac{x_2}{\alpha}$ , and  $\sum_{i=3}^{\infty} y_i \geq \sum_{i=3}^N 2^{-(i-2)} v_i$  because the rest of the  $y_i$  for  $i > N$  are nonnegative. We also have

$$\sum_{i=3}^{\infty} 2^{-(i-2)} y_i \leq \sum_{i=3}^{\infty} 2^{-(i-2)} \cdot 1 = 1$$

and so  $y_1 \leq 1$  because  $x_2 \geq 0$ . Then  $x_2 \geq 0$  also implies  $y_2 \geq 0$ , and as  $x_2 \leq \frac{1}{6}$ , we have

$$y_2 = \frac{x_2}{\alpha} \leq \frac{1}{6} \cdot \frac{12}{5} = \frac{2}{5} \leq 1.$$

Then  $z_1 \in [0, 1]$  because it is a sum of nonnegative numbers and

$$z_1 = \sum_{i=3}^{\infty} 2^{-(i-2)} z_i \leq \sum_{i=3}^{\infty} 2^{-(i-2)} \cdot 1 = 1.$$

We have  $z_2 \in [0, 1]$  because it is 0. We have therefore shown that  $y_i, z_i \geq 0$  and  $\|(y_i)\| = \|(z_i)\| = 1$ . Therefore, to complete the proof that  $(x_i), (y_i) \in K$ , we only need to show  $y_1 + y_2 = \sum_{i=3}^{\infty} 2^{-(i-2)} y_i$  and likewise for  $(z_i)$ , and this follows trivially from the definitions in each case.

We can now show  $\alpha y_i - (1 - \alpha) z_i = x_i$ . For  $3 \leq i \leq N$ , we have already shown  $\alpha v_i - (1 - \alpha) \zeta_i = x_i$ . For  $i > N$  and  $x_i \geq 0$ , we have  $\alpha \frac{x_i}{\alpha} - (1 - \alpha) \cdot 0 = x_i$ . For  $i > N$  and  $x_i \leq 0$ , we have  $\alpha 0 - (1 - \alpha) \frac{-x_i}{1 - \alpha} = x_i$  too. For  $i = 0$ , we have the fact that  $\alpha \cdot 1 - (1 - \alpha) \cdot 1 = 2\alpha - 1 = x_0$ . For  $i = 2$  we have  $\alpha \frac{x_2}{\alpha} - (1 - \alpha) \cdot 0 = x_2$ . Finally, for  $i = 1$  we have

$$\begin{aligned} & \alpha \left( \sum_{i=3}^{\infty} 2^{-(i-2)} y_i - \frac{x_2}{\alpha} \right) - (1 - \alpha) \sum_{i=3}^{\infty} 2^{-(i-2)} z_i \\ &= -x_2 + \sum_{i=3}^{\infty} 2^{-(i-2)} (\alpha y_i - (1 - \alpha) z_i) \\ &= -x_2 + \sum_{i=3}^{\infty} 2^{-(i-2)} x_i = x_1 \end{aligned}$$

because  $(x_i) \in E$ .

– If  $x_2 \leq 0$ :

Define  $x'_i = -x_i$  and apply the previous case to obtain  $\alpha' \in [0, 1]$  and  $(y'_i), (z'_i) \in K$  such that  $\alpha' y'_i - (1 - \alpha') z'_i = x'_i$ . Then define  $\alpha = (1 - \alpha')$ ,  $y_i = z'_i$  and  $z_i = y'_i$  and we have  $x_i = \alpha y_i - (1 - \alpha) z_i$ .

- $K$  complete in  $\|\cdot\|_{c_0}$ :

Let  $x_i$  be a Cauchy sequence of elements of  $K$  in the  $c_0$ -norm. As  $E$  is complete, there exists some  $y \in E$  such that  $x_i \rightarrow y$ . In particular, we have that for all  $j \in \mathbb{N}$ ,  $x_{ij} \rightarrow y_j$  in the usual topology of real numbers. As each  $x_{i0} = 1$ , we have  $y_0 = 1$ . For each  $j \in \mathbb{N}$ , we have  $x_{ij} \geq 0$ , so  $y_j \geq 0$ . Any norm is continuous in the topology it defines, so we also have  $\|y\| = \lim_{i \rightarrow \infty} \|x_i\| = 1$ , and therefore  $y \in K$ , and so  $K$  is complete.

- $E = F$  and  $(E, E_+, \tau)$  a Banach base-norm space:

By definition,  $F \subseteq E$ . If  $x \in E$ , then

$$x \in \|x\|_{c_0} \text{Ball}(\|\cdot\|_{c_0}) \subseteq \|x\|_{c_0} 6\text{absco}(K) \subseteq F.$$

Therefore  $F = E$ , so  $(E, E_+, \tau)$  is a Banach pre-base-norm space. By Lemma 2.2.14,  $E_+$  is complete in the  $c_0$ -norm, which is equivalent to the base-norm, therefore  $E_+$  is closed in the base-norm, and so  $(E, E_+, \tau)$  is a Banach base-norm space.  $\square$

Now define  $x = (0, \frac{1}{2}, -\frac{1}{2}, 0, \dots) \in E$ .

**Counterexample A.6.2.** *The point  $x \notin \text{absco}(K)$ , but for all  $\alpha \in [0, 1]$ ,  $\alpha x \in \text{absco}(K)$ . Therefore  $(E, E_+, \tau)$  is a Banach base-norm space that is not radially compact.*

*Proof.* The first part of the proof is to show that  $x \notin \text{absco}(K)$ . Recall that, by Lemma 0.1.1, we want to show that there can be no  $\alpha \in [0, 1]$ ,  $(y_i), (z_i) \in K$  such that  $\alpha y_i - (1 - \alpha)z_i = x_i$ . So assume for a contradiction that such a decomposition of  $x$  exists. Then

$$0 = x_0 = \alpha y_0 - (1 - \alpha)z_0 = 2\alpha - 1,$$

and so we must have  $\alpha = \frac{1}{2}$ . We have

$$\begin{aligned} \frac{1}{2}y_1 - \frac{1}{2}z_1 &= \frac{1}{2} \\ \frac{1}{2}y_2 - \frac{1}{2}z_2 &= -\frac{1}{2}, \end{aligned}$$

clearing the denominators gives

$$\begin{aligned} y_1 - z_1 &= 1 \\ y_2 - z_2 &= -1. \end{aligned}$$

The only way to satisfy these conditions and have  $x_i, y_i \in [0, 1]$  is to take  $y_1 = 1, z_1 = 0$  and  $y_2 = 0, z_2 = 1$ . We require

$$1 + 0 = \sum_{i=3}^{\infty} 2^{-(i-2)} y_i.$$

For any sequence  $w_i \in [0, 1]$ , we have  $\sum_{i=3}^{\infty} 2^{-(i-2)} w_i \in [0, 1]$ , because  $[0, 1]$  is complete and convex. Since  $y_i \rightarrow 0$ , there is some  $k$  such that  $y_i < 1$ . We can define  $y'_i = y_i$  for  $i \neq k$  and  $y'_k = 1$ . Then

$$\sum_{i=3}^{\infty} 2^{-(i-2)} y_i < \sum_{i=3}^{\infty} 2^{-(i-2)} y'_i \leq 1,$$

so we cannot obtain a  $y_i \rightarrow 0$  where  $\sum_{i=3}^{\infty} 2^{-(i-2)} y_i = 1 = y_1 + y_2$ . Therefore  $x_i \notin \text{absco}(K)$ .

We have  $0 \cdot x = 0 \in \text{absco}(K)$  because every absolutely convex set contains 0. We will now show that  $\beta x \in \text{absco}(K)$  for every  $\beta \in (0, 1)$ . We have  $\beta x = (0, \frac{1}{2}\beta, -\frac{1}{2}\beta, 0, \dots)$ . Take the largest  $N \in \mathbb{N}$  (possibly 0) such that  $1 - 2^{-N} \leq \beta$ . We therefore have

$$1 - 2^{-N} \leq \beta \leq 1 - 2^{-(N+1)},$$

so

$$\begin{aligned} 0 &= (1 - 2^{-N}) - (1 - 2^{-N}) \leq \alpha - (1 - 2^{-N}) \leq (1 - 2^{-(N+1)}) - (1 - 2^{-N}) \\ &= 2^{-(N+1)}. \end{aligned}$$

We therefore have  $0 \leq 2^{N+1}(\alpha - (1 - 2^{-N})) \leq 1$ . We define

$$y_i = \begin{cases} 1 & \text{if } 3 \leq i \leq N+2 \\ 2^{N+1}(\alpha - (1 - 2^{-N})) & \text{if } i = N+3 \\ 0 & \text{if } i \geq N+4. \end{cases}$$

By the preceding arguments,  $y_i \in [0, 1]$  and as it is eventually 0,  $y_i \rightarrow 0$ . We define  $y_0 = 1$ ,  $y_1 = \alpha$  and  $y_2 = 0$ . We then have

$$\begin{aligned} \sum_{i=3}^{\infty} 2^{-(i-2)} y_i &= \sum_{i=3}^{N+2} 2^{-(i-2)} + 2^{-(N+3-2)} 2^{N+1}(\alpha - (1 - 2^{-N})) + 0 \\ &= 1 - 2^{-N} + \alpha - (1 - 2^{-N}) \\ &= \alpha \\ &= y_1 + y_2, \end{aligned}$$

so  $(y_i) \in K$ . We can define  $z_0 = 1$ ,  $z_1 = 0$ ,  $z_2 = \alpha$  and  $z_i = y_i$  for  $i \geq 3$ . Then similarly to the argument above, we have  $(z_i) \in K$ . Taking  $\alpha = \frac{1}{2}$ , we have that for  $i = 0$ :

$$\frac{1}{2}y_0 - \frac{1}{2}z_0 = \frac{1}{2} - \frac{1}{2} = 0 = \alpha x_0.$$

For  $i = 1$ :

$$\frac{1}{2}y_1 - \frac{1}{2}z_1 = \frac{1}{2}\alpha = \alpha x_1,$$

and for  $i = 2$ :

$$\frac{1}{2}y_2 - \frac{1}{2}z_2 = -\frac{1}{2}\alpha = \alpha x_2.$$



For  $i \geq 3$ , we have  $z_i = y_i$  so

$$\frac{1}{2}y_i - \frac{1}{2}z_i = 0 = \alpha x_i,$$

so we have shown that  $\alpha(x_i) \in \text{absco}(K)$ .

By absolute convexity, we also have  $-\alpha x \in \text{absco}(K)$ , so for all  $\alpha \in (-1, 1)$ , we have  $\alpha x \in \text{absco}(K)$ , and for all other  $\alpha$  we have  $\alpha x \notin \text{absco}(K)$ , because otherwise we could use the convexity of  $\text{absco}(K)$  to prove  $x \in \text{absco}(K)$ , which is false. We therefore have that the intersection of the ray generated by  $x$  with  $\text{absco}(K)$  is homeomorphic to  $(-1, 1)$  and therefore not compact, so  $\text{absco}(K)$  is not radially compact.  $\square$



# Appendix B

## Summary

In general, we study pairs of categories, where one consists of state spaces and state transformers, the other of algebras of predicates and predicate transformers, and there is a contravariant equivalence of categories between the two.

Another common thread is the use of probability monads, a categorical way of representing probabilistic maps (by using the Kleisli category) and convex sets (by using the Eilenberg-Moore category). The monads  $\mathcal{D}$  and  $\mathcal{D}_\infty$  are known as distribution monads. We can describe  $\mathcal{D}$  as mapping a set to the set of discrete probability distributions of finite support on it, and  $\mathcal{D}_\infty$  as mapping a set to the set of discrete probability distributions of countable support on it. We also consider  $\mathcal{E}$ , the expectation monad, one version of which maps a set  $X$  to the finitely-additive measures on  $\mathcal{P}(X)$ . On compact Hausdorff spaces, we use the Radon monad  $\mathcal{R}$ , which assigns to a compact Hausdorff space  $X$  the space of Radon probability measures on it, or equivalently the state space of the  $C^*$ -algebra  $C(X)$ .

In the first chapter, we describe a probabilistic version of Gelfand duality, where, in the category of commutative  $C^*$ -algebras, we replace  $*$ -homomorphisms with maps that are only required to preserve positive elements and the unit, and we replace the category of compact Hausdorff spaces with the Kleisli category of the Radon monad. Kleisli categories of probability monads are a standard way of producing a category of probabilistic mappings.

If we consider non-commutative  $C^*$ -algebras, a natural category to embed them in, when considering only the order structure, is order-unit spaces. If we consider the dual spaces of  $C^*$ -algebras, or alternatively the preduals of  $W^*$ -algebras, a natural category to embed them in, when considering only the

order structure, is base-norm spaces. The definition of an order-unit space is stable, *i.e.* apparently different definitions used by various authors are equivalent. However, this is not the case for base-norm spaces.

We give three definitions of base-norm space and examples distinguishing them. We then show that each bounded convex set defines a pre-base-norm space, which is a Banach base-norm space iff the original convex set is sequentially complete. Later, we show that bases of Banach base-norm spaces, equivalently sequentially complete convex sets, are a reflective subcategory of both the categories of Eilenberg-Moore algebras  $\mathcal{EM}(\mathcal{D})$  and  $\mathcal{EM}(\mathcal{D}_\infty)$ .

We show that taking the dual space defines a dual adjunction between pre-base-norm spaces and order-unit spaces, and that this adjunction restricts to an equivalence between reflexive spaces, such as finite-dimensional spaces.

To extend this duality to a larger class of spaces including all  $C^*$ -algebras, we first describe the duality between Banach spaces and Smith spaces. Smith spaces are a kind of space, originated by Akbarov, that characterize the “bounded weak- $*$ ” topology on the dual of a Banach space. We can then define Smith base-norm spaces, which are dual to Banach order-unit spaces, and Smith order-unit spaces, which are dual to Banach base-norm spaces. We can also combine these dualities with the previous adjunction to show that the double dual space is an “enveloping” Smith space, analogous to the enveloping  $W^*$ -algebra of a  $C^*$ -algebra.

We go over Świrszcz’s theorem that  $\mathcal{EM}(\mathcal{R})$  and  $\mathcal{EM}(\mathcal{E})$  are equivalent to the category of compact convex subsets of locally convex topological vector spaces. This gives us a characterization of the bases of Smith base-norm spaces and the unit intervals of Smith order-unit spaces without having to consider an embedding in a vector space.

# Appendix C

## Samenvatting

In het algemeen, bestuderen we paren van categorieën, waarvan de ene uit toestandruimtes en transformaties van toestanden bestaat, en de andere uit algebra's van predicaten en transformaties van predicaten bestaat, met een equivalentie van categorieën ertussen.

Een andere algemene rode draad is het gebruik van kansmonaden. Die geven een categorische manier om stochastieke afbeeldingen (met behulp van de Kleisli categorie) en convexe verzamelingen (met behulp van de Eilenberg-Moore categorie) te representeren. De monaden  $\mathcal{D}$  en  $\mathcal{D}_\infty$  heten kansverdeling-monaden. De monade  $\mathcal{D}$  beeldt een verzameling af op de verzameling van discrete kansverdelingen met eindige drager, en de monade  $\mathcal{D}_\infty$  beeldt een verzameling af op de verzameling van discrete kansverdelingen met aftelbare drager. We beschouwen ook  $\mathcal{E}$ , de verwachtingmonade, waarvan één uitvoering een verzameling  $X$  afbeeldt op de verzameling van eindig additieve kansmaten op  $\mathcal{P}(X)$ . Op compacte Hausdorff ruimtes gebruiken we de Radonmonade die een compacte Hausdorff ruimte  $X$  afbeeldt op de ruimte van Radon kansmaten op  $X$ , of equivalent, op de toestandruimte van de  $C^*$ -algebra  $C(X)$ .

In het eerste hoofdstuk, beschrijven we een stochastische versie van Gelfand-dualiteit, waarbij we geen  $*$ -homomorfismen in de categorie van  $C^*$ -algebra's gebruiken maar lineaire afbeeldingen die positieve elementen en de eenheid bewaren, en waarbij we niet de categorie van compacte Hausdorff ruimtes gebruiken, maar de Kleisli categorie van deze Radonmonad. Kleisli categorieën zijn een standaard manier om een categorie van stochastische afbeeldingen te maken.

Voor het bestuderen van de structuur van de ordening van niet-commutatieve  $C^*$ -algebras gebruiken we een natuurlijke inbedding in de categorie van

orde-eenheidsruimtes. Op vergelijkbare wijze bestuderen we de structuur van de ordening van de dualen van  $C^*$ -algebra's, of de predualen van  $W^*$ -algebra's, via een natuurlijke inbedding in de categorie van basisnormruimtes. De definitie van een orde-eenheidsruimte is stabiel, *d.w.z.* ogenschijnlijk ongelijke definities die verschillende auteurs gebruiken zijn equivalent. Dit geldt niet voor basisnormruimtes.

We geven drie verschillende definities van het begrip basisnormruimte en we geven voorbeelden die hen onderscheiden. Dan bewijzen we dat elke begrensde convexe verzameling een prebasisnormruimte definieert, die een Banachbasisnormruimte is dan en slechts dan als de oorspronkelijke convexe verzameling sequentiël volledig is. Tenslotte bewijzen we dat basissen van Banachbasisnormruimtes, of sequentiël volledige convexe verzamelingen, een reflectieve deelcategorie van de categorieën van Eilenberg-Moore algebras  $\mathcal{EM}(\mathcal{D})$  en  $\mathcal{EM}(\mathcal{D}_\infty)$  vormen.

We tonen aan dat afbeelden naar de duale ruimte een contravariante adjunctie tussen prebasisnormruimtes en orde-eenheidsruimtes definiëren, en dat deze adjunctie zich beperkt tot een equivalentie tussen reflexieve ruimtes, bijvoorbeeld eindigdimensionale ruimtes.

Om deze dualiteit tot een grotere klasse van ruimtes uit te breiden, beschrijven we eerst de dualiteit tussen Banachruimtes en Smithruimtes. Smithruimtes vormen een soort ruimte die oorspronkelijk gedefinieerd zijn door Akbarov en die de “begrensde zwakke ster” topologie op de duale ruimte van een Banachruimte karakteriseren. Daarmee kunnen we Smithbasisnormruimtes definiëren, die de duale van Banach-orde-eenheidsruimtes zijn, en Smith-orde-eenheidsruimtes, die de duale van Banachbasisnormruimtes zijn. We kunnen ook deze dualiteiten met de vorige adjunctie samenvoegen om te laten zien dat de dubbel duale ruimte een “omhullende” Smithruimte is, zoals de dubbel duale ruimte van een  $C^*$ -algebra de omhullende  $W^*$ -algebra is.

We geven een samenvatting van de stelling van Świrszcz, die zegt dat  $\mathcal{EM}(\mathcal{R})$  en  $\mathcal{EM}(\mathcal{E})$  equivalent zijn aan de categorie van compacte convexe deelverzamelingen van lokaal convexe topologische vectorruimtes. Deze stelling geeft ons een karakterisering van de basissen van Smithbasisnormruimtes en van de eenheidsintervallen van Smith-orde-eenheidsruimtes, zonder gebruik van een inbedding in een vectorruimte.

# Appendix D

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# Bibliography

- [1] Samson Abramsky and Bob Coecke. A Categorical Semantics of Quantum Protocols. In K. Engesser, Dov M. Gabbai, and D. Lehmann, editors, *Handbook of Quantum Logic and Quantum Structures*, pages 261–323. North Holland, Elsevier, Computer Science Press, 2009. 44
- [2] Sergei S. Akbarov. Pontryagin Duality in the Theory of Topological Vector Spaces and in Topological Algebra. *Journal of Mathematical Sciences*, 113(2):179–349, 2003. 145, 146, 162, 165
- [3] Sergei S. Akbarov. Holomorphic Functions of Exponential Type and Duality for Stein Groups with Algebraic Connected Component of Identity. *Journal of Mathematical Sciences*, 162(4):459–586, 2009. 145
- [4] Erik M. Alfsen. *Compact Convex Sets and Boundary Integrals*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1971. 91, 101, 103, 137, 175, 181, 213
- [5] Erik M. Alfsen, Harald Hanche-Olsen, and Frederic W. Shultz. State Spaces of  $C^*$ -algebras. *Acta Mathematica*, 144(1):267–305, 1980. 204
- [6] Erik M. Alfsen and Frederic W. Shultz. *State Spaces of Operator Algebras*. Birkhäuser, 2001. 56, 91, 101, 103, 106, 134, 137, 232
- [7] Erik M. Alfsen and Frederic W. Shultz. *Geometry of State Spaces of Operator Algebras*. Birkhäuser, 2003. 143
- [8] Leonard Asimow and Alan J. Ellis. *Convexity Theory and its Applications in Functional Analysis*. Number 16 in L.M.S. Monographs. Academic Press, 1980. 101, 148, 163
- [9] Howard Barnum and Alexander Wilce. Ordered Linear Spaces and Categories as Frameworks for Information-Processing Characterizations of

- Quantum and Classical Theory. <https://arxiv.org/abs/0908.2354>, 2009. 4, 88
- [10] Michael Barr and Charles Wells. Toposes, Triples and Theories. *Reprints in Theory and Applications of Categories*, 12:1–288, 2005. 31, 32
  - [11] Jonathan Barrett. Information Processing in Generalized Probabilistic Theories. <https://arxiv.org/abs/quant-ph/0508211>, 2006. 4, 88
  - [12] Jonathan Barrett. Information Processing in Generalized Probabilistic Theories. *Physical Review A*, 75:032304, Mar 2007. 4, 88
  - [13] Patrick Billingsley. *Convergence of Probability Measures*. John Wiley and Sons, 1968. 64
  - [14] Francis Borceux. *Handbook of Categorical Algebra*, volume 50, 51 and 52 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 1994. 29, 31, 34, 41
  - [15] Nicolas Bourbaki. *Topological Vector Spaces*. Springer, 1987. 23, 24, 25, 26, 157
  - [16] Nicolas Bourbaki. *General Topology*. Ettore Majorana International Science. Springer, 1998. 18, 153, 155, 157, 160, 201
  - [17] Kenta Cho. Semantics for a Quantum Programming Language by Operator Algebras. *New Generation Computing*, 34(1):25–68, 2016. 5
  - [18] Kai Lai Chung. *Lectures from Markov Processes to Brownian Motion*. Number 249 in Grundlehren der mathematischen Wissenschaften. Springer, 1982. 82
  - [19] Johann Cigler, Viktor Losert, and Peter Michor. *Banach Modules and Functors on Categories of Banach Spaces*, volume 46 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker Inc., 1979. 146
  - [20] John B. Conway. *A Course In Functional Analysis, Second Edition*, volume 96 of *Graduate Texts in Mathematics*. Springer Verlag, 1990. 8, 12, 13, 16, 26, 69, 78, 110, 142, 166, 211, 217, 235
  - [21] James B. Cooper. *Saks Spaces and Applications to Functional Analysis*, volume 28 of *North-Holland Mathematics Studies*. North-Holland Publishing Company, 1978. 165



- [22] John Dauns. Categorical  $W^*$ -Tensor Product. *Transactions of the American Mathematical Society*, 166:pp. 439–456, 1972. 201
- [23] E. Brian Davies and John T. Lewis. An Operational Approach to Quantum Probability. *Comm. Math. Phys.*, 17(3):239–260, 1970. 4, 87
- [24] Ellie d’Hondt and Prakash Panangaden. Quantum Weakest Preconditions. *Mathematical Structures in Computer Science*, 16(3):429–451, 2006. 5
- [25] Jacques Dixmier. Sur un théorème de Banach. *Duke Mathematical Journal*, 15(4):1057–1071, 12 1948. 165
- [26] Jacques Dixmier.  *$C^*$ -algebras*, volume 15 of *North-Holland Mathematical Library*. North-Holland Publishing Company, 1977. 47, 49, 56, 59, 67, 105, 205
- [27] Joseph L. Doob. *Stochastic Processes*. John Wiley and Sons, 1953. 71
- [28] Nelson Dunford and Jacob T. Schwartz. *Linear Operators, Part 1: General Theory*. Number VII in Pure and Applied Mathematics. Interscience Publishers, 1957. 147, 152, 163, 165, 166, 235
- [29] Anatolij Dvurečenskij and Sylvia Pulmannová. *New Trends in Quantum Structures*. Kluwer Acad. Publ., Dordrecht, 2000. 53
- [30] Eugene B. Dynkin. *Markov Processes*. Number 121 in Grundlehren der mathematischen Wissenschaften. Springer, 1965. 82
- [31] Christopher M. Edwards. The Operational Approach to Algebraic Quantum Theory I. *Communications in Mathematical Physics*, 16(3):207–230, 1970. 4, 56, 87, 88, 106, 107
- [32] Christopher M. Edwards and Michael A. Gerzon. Monotone Convergence in Partially Ordered Vector Spaces. *Annales de l’Institut Henri Poincaré A*, 12:323–328, 1970. 124
- [33] David A. Edwards. On the Homeomorphic Affine Embedding of a Locally Compact Cone into a Banach Dual Space Endowed with the Vague Topology. *Proceedings of the London Mathematical Society*, 14:399–414, 1964. 174
- [34] Alan J. Ellis. The Duality of Partially Ordered Normed Linear Spaces. *Journal of the London Mathematical Society*, s1-39(1):730–744, 1964. 137, 147, 182, 192

- [35] Alan J. Ellis. Minimal Decompositions in Base Normed Spaces. In Alois Hartkämper and Hans Neumann, editors, *Foundations of Quantum Mechanics and Ordered Linear Spaces: Advanced Study Institute Marburg 1973*, pages 30–32. Springer Berlin Heidelberg, 1974. 104, 235
- [36] Ryszard Engelking. *General Topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag Berlin, 1989. 76
- [37] Vitaly V. Fedorchuk. Probability Measures in Topology. *Russian Mathematical Surveys*, 46(1):45–93, 1991. 64, 213
- [38] Willy Feller. On the Integro-Differential Equations of Purely Discontinuous Markoff Processes. *Transactions of the American Mathematical Society*, 48:488–515, 1940. 71, 82
- [39] Willy Feller. *An Introduction to Probability Theory and its Applications, Volume II*. John Wiley and Sons, 1966. 82
- [40] David J. Foulis and Mary K. Bennett. Effect Algebras and Unsharp Quantum Logics. *Foundations of Physics*, 24(10):1331–1352, 1994. 53
- [41] David H. Fremlin. Measure Theory, Volume 4. <https://www.essex.ac.uk/maths/people/fremlin/mt.htm>, 2003. 64, 76, 77
- [42] Tobias Fritz. A Presentation of the Category of Stochastic Matrices. <https://arxiv.org/abs/0902.2554>, 2009. 63
- [43] Robert Furber and Bart Jacobs. From Kleisli Categories to Commutative C\*-algebras: Probabilistic Gelfand Duality. In Reiko Heckel and Stefan Milius, editors, *Algebra and Coalgebra in Computer Science*, volume 8089 of *Lecture Notes in Computer Science*, pages 141–157. Springer Berlin Heidelberg, 2013. 43
- [44] Robert Furber and Bart Jacobs. From Kleisli Categories to Commutative C\*-algebras: Probabilistic Gelfand Duality. *Logical Methods in Computer Science*, 11(2), 2014. 6, 43, 203, 209, 213
- [45] Robert Furber, Mathys Rennela, and Sam Staton. Infinite-Dimensionality in Quantum Foundations: W\*-algebras as Presheaves over Matrix Algebras. *Electronic Proceedings in Theoretical Computer Science*, 236:161–173, 2017. 48

- [46] Michèle Giry. A Categorical Approach to Probability Theory. In B. Banaschewski, editor, *Categorical Aspects of Topology and Analysis*, pages 68–85. Springer, 1980. 45, 76
- [47] Stanley Gudder. Convex Structures and Operational Quantum Mechanics. *Communications in Mathematical Physics*, 29(3):249–264, 1973. 4, 87, 88, 96, 98, 129
- [48] Stanley Gudder and Sylvia Pulmannová. Representation Theorem for Convex Effect Algebras. *Commentationes Mathematicae Universitatis Carolinae*, 39(4):645 – 659, 1998. 52, 53, 147
- [49] Paul R. Halmos. *Measure Theory*, volume 18 of *Graduate Texts in Mathematics*. Springer, 1950. 64, 76
- [50] Teiko Heinosaari and Mario Ziman. *The Mathematical Language of Quantum Theory. From Uncertainty to Entanglement*. Cambridge University Press, 2012. 53
- [51] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. Separability of Mixed States: Necessary and Sufficient Conditions. *Physics Letters A*, 223(12):1 – 8, 1996. 48
- [52] Bart Jacobs. Convexity, Duality and Effects. In Cristian S. Calude and Vladimiro Sassone, editors, *Theoretical Computer Science*, volume 323 of *IFIP Advances in Information and Communication Technology*, pages 1–19. Springer Berlin Heidelberg, 2010. 3, 40, 52, 88, 137
- [53] Bart Jacobs. Involutive Categories and Monoids, with a GNS-correspondence. *Foundations of Physics*, 42(7):874–895, 2012. 46
- [54] Bart Jacobs. Measurable Spaces and their Effect Logic. In *Logic in Computer Science*. IEEE, Computer Science Press, 2013. 210
- [55] Bart Jacobs. Lectures on Kadison Duality. <https://www.cs.ru.nl/B.Jacobs/TALKS/kadison-lectures-6up.pdf>, 2014. 147, 181
- [56] Bart Jacobs. A Recipe for State-and-Effect Triangles. In Lawrence S. Moss and Paweł Sobociński, editors, *6th Conference on Algebra and Coalgebra in Computer Science (CALCO 2015)*, volume 35 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 116–129, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. 3

- [57] Bart Jacobs. New Directions in Categorical Logic, for Classical, Probabilistic and Quantum Logic. *Logical Methods in Computer Science*, 11(3), 2015. 3
- [58] Bart Jacobs and Jorik Mandemaker. Coreflections in Algebraic Quantum Logic. *Foundations of Physics*, 42(7):932–958, 2012. 52, 53, 54
- [59] Bart Jacobs and Jorik Mandemaker. The Expectation Monad in Quantum Foundations. In Bart Jacobs, Peter Selinger, and Bas Spitters, editors, *Quantum Physics and Logic (QPL) 2011*, volume 95 of *Electronic Proceedings in Theoretical Computer Science*, pages 143–182, 2012. 52, 54, 55, 56, 61, 69, 70, 88, 147, 181, 209, 223
- [60] Bart Jacobs, Jorik Mandemaker, and Robert Furber. The Expectation Monad in Quantum Foundations. *Information and Computation*, 250:87–114, 2016. 6, 7, 43, 209, 217
- [61] Graham Jameson. *Ordered Linear Spaces*, volume 141 of *Lecture Notes in Mathematics*. Springer, 1970. 55
- [62] Peter Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982. 41, 51
- [63] Richard Kadison and John Ringrose. *Fundamentals of the Theory of Operator Algebras*. Academic Press, 1983. 47, 49, 204, 205
- [64] Richard V. Kadison. *A Representation Theory for Commutative Topological Algebra*, volume 7 of *Memoirs of the AMS*. AMS, 1951. 147, 181
- [65] Richard V. Kadison. Order Properties of Bounded Self-Adjoint Operators. *Proceedings of the American Mathematical Society*, 2(3):505–510, 1951. 57
- [66] Richard V. Kadison. Transformations of States in Operator Theory and Dynamics. *Topology*, 3, Supplement 2(0):177 – 198, 1965. 181
- [67] Nigel J. Kalton and N. Tenney Peck. A Re-examination of the Roberts Example of a Compact Convex Set without Extreme Points. *Mathematische Annalen*, 253(2):89–101, 1980. 217
- [68] Klaus Keimel. Abstract Ordered Compact Convex Sets and Algebras of the (Sub)Probabilistic Powerdomain Monad over Ordered Compact Spaces. *Algebra and Logic*, 48(5):330, 2009. 66, 217

- [69] John L. Kelley. *General Topology*. Number 27 in Graduate Texts in Mathematics. Springer, 1955. 162, 229
- [70] Heinrich Kleisli. Every Standard Construction is Induced by a Pair of Adjoint Functors. *Proceedings of the American Mathematical Society*, 16(3):544–546, 1965. 76
- [71] Anthony W. Knap. *Representation Theory of Semisimple Groups*. Princeton University Press, 1986. 143
- [72] Andrey N. Kolmogorov. Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. *Mathematische Annalen*, 104:415–458, 1931. 71
- [73] Heinz König. Theory and Applications of Superconvex Spaces. In Rainer Nagel, Ulf Schlotterbeck, and Manfred P. H. Wolff, editors, *Aspects of Positivity in Functional Analysis*, volume 122 of *North-Holland Mathematics Studies*, pages 79–118. North-Holland, 1986. 107
- [74] Heinz König and Gerd Wittstock. Superconvex Sets and  $\sigma$ -convex Sets, and the Embedding of Convex and Superconvex Spaces. *Note di Matematica*, 10(Suppl. 2):343–362, 1990. 107
- [75] Dexter Kozen. A Probabilistic PDL. *Journal of Computer and System Sciences*, 30(2):162 – 178, 1985. 45, 81
- [76] Joachim Lambek and Philip Scott. *Introduction to Higher Order Categorical Logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1986. 142
- [77] Klaas P. Landsman. *Mathematical Topics Between Classical and Quantum Mechanics*. Springer Monographs in Mathematics. Springer, 1998. 204
- [78] William Lawvere. The Category of Probabilistic Mappings, 1962. 71, 76
- [79] Tom Leinster. *Higher Operads, Higher Categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2004. 34, 35
- [80] Hans Maassen. Quantum Probability and Quantum Information Theory. In Fabio Benatti, Mark Fannes, Roberto Floreanini, and Dimitri Petritis, editors, *Quantum Information, Computation and Cryptography*, number 808 in *Lecture Notes in Physics*, pages 65–108. Springer, Berlin, 2010. 63

- [81] Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer Verlag, 1971. 29, 30, 31, 33, 41, 162
- [82] George W. Mackey. A Theorem of Stone and von Neumann. *Duke Mathematical Journal*, 16(2):313–326, 6 1949. 143
- [83] Ernest G. Manes. A Triple-Theoretic Construction of Compact Algebras. In Beno Eckmann, editor, *Seminar on Triples and Categorical Homology Theory*, number 80 in Lecture Notes in Mathematics, pages 91–118. Springer, Berlin, 1969. 41, 50
- [84] Ernest G. Manes. *Algebraic Theories*, volume 26 of *Graduate Texts in Mathematics*. Springer, 1974. 31
- [85] Michael Mislove. Probabilistic Monads, Domains and Classical Information. In Elham Kashefi, Jean Krivine, and Femke van Raamsdonk, editors, Proceedings 7th International Workshop on *Developments of Computational Methods*, Zurich, Switzerland, 3rd July 2011, volume 88 of *Electronic Proceedings in Theoretical Computer Science*, pages 87–100. Open Publishing Association, 2012. 48, 64
- [86] Michael Mislove, Joel Ouaknine, Duško Pavlović, and James Worrell. Duality for Labelled Markov Processes. In Igor Walukiewicz, editor, *Foundations of Software Science and Computation Structures*, volume 2987 of *Lecture Notes in Computer Science*, pages 393–407. Springer Berlin Heidelberg, 2004. 44
- [87] Eugenio Moggi. Notions of Computation and Monads. *Information and Computation*, 93(1):55–92, 1991. 32
- [88] Rainer J. Nagel. Order Unit and Base Norm Spaces. In Alois Hartkämper and Hans Neumann, editors, *Foundations of Quantum Mechanics and Ordered Linear Spaces*, volume 29 of *Lecture Notes in Physics*, pages 23–29. Springer Berlin Heidelberg, 1974. 101
- [89] Isaac Namioka and Robert R. Phelps. Tensor Products of Compact Convex Sets. *Pacific Journal of Mathematics*, 31(2):469–480, 1969. 88
- [90] Kung-fu Ng. On a Theorem of Dixmier. *Mathematica Scandinavica*, 29:279–280, 1971. 165

- [91] Masanao Ozawa. Optimal Measurements for General Quantum Systems. *Reports on Mathematical Physics*, 18:11–28, 1980. 4, 5, 85, 88, 130, 134, 192
- [92] Prakash Panangaden. *Labelled Markov Processes*. World Scientific, 2009. 71, 74, 81
- [93] Joan Wick Pelletier and Jiří Rosický. On the Equational Theory of  $C^*$ -algebras. *Algebra Universalis*, 30:275–284, 1993. 44
- [94] Robert R. Phelps. *Lectures on Choquet’s Theorem*, volume 1757 of *Lecture Notes in Mathematics*. Springer, 2001. 213
- [95] Albrecht Pietsch. *Nuclear Locally Convex Spaces*. Number 66 in *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer Verlag, 1972. 14
- [96] Gordon D. Plotkin. Dijkstra’s Predicate Transformers and Smyth’s Powerdomains. In Dines Bjørner, editor, *Abstract Software Specifications: 1979 Copenhagen Winter School January 22 – February 2, 1979 Proceedings*, pages 527–553. Springer Berlin Heidelberg, Berlin, Heidelberg, 1980. 3
- [97] Dieter Pumplün. The Universal Compactification of Topological Convex Sets and Modules. *Journal of Convex Analysis*, 8(1):255–268, 2001. 202
- [98] Dieter Pumplün. The Metric Completion of Convex Sets and Modules. *Results in Mathematics*, 41(3-4):346–360, 2002. 123, 124, 192
- [99] Dieter Pumplün and Helmut Röhrl. Banach Spaces and Totally Convex Spaces I. *Communications in Algebra*, 12(8):953–1019, 1984. 8, 114
- [100] Dieter Pumplün and Helmut Röhrl. Banach Spaces and Totally Convex Spaces II. *Communications in Algebra*, 13(5):1047–1113, 1985. 8
- [101] Mathys Rennela. Operator Algebras in Quantum Computation. Master’s thesis, Université Paris 7 Diderot, 2013. <https://arxiv.org/abs/1510.06649>. 5, 207
- [102] Mathys Rennela and Sam Staton. Complete Positivity and Natural Representation of Quantum Computations. *Electronic Notes in Theoretical Computer Science*, 319:369 – 385, 2015. The 31st Conference on the Mathematical Foundations of Programming Semantics (MFPS XXXI). 207

- [103] James W. Roberts. A Compact Convex Set with no Extreme Points. *Studia Mathematica*, 60(3):255–266, 1977. 217
- [104] Gerd Rodé. Superkonvexität und schwache Kompaktheit. *Archiv der Mathematik*, 36(1):62–72, 1981. 107
- [105] Stefan Rolewicz. *Metric Linear Spaces*. Mathematics and its Applications. D. Reidel Publishing Company, 1984. 217
- [106] Halsey L. Royden. *Real Analysis*. The Macmillan Company, 1963. 74, 76, 77
- [107] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, 1987. Third, International edition. 45, 64
- [108] Bernard Russo and Henry A. Dye. A Note on Unitary Operators in  $C^*$ -algebras. *Duke Math. J.*, 33:413–416, 1966. 50
- [109] Shôichirô Sakai.  *$C^*$ -algebras and  $W^*$ -algebras*, volume 60 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1971. 49, 105
- [110] Helmut H. Schaefer. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer Verlag, 1966. 7, 8, 10, 11, 12, 14, 15, 16, 18, 23, 24, 25, 26, 99, 102, 159, 160, 161, 162, 166, 211
- [111] René L. Schilling. *Measures, Integrals and Martingales*. Cambridge University Press, 2005. 73, 74
- [112] Zbigniew Semadeni. *Monads and their Eilenberg-Moore Algebras in Functional Analysis*, volume 33 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University at Kingston, Ontario, Canada, 1973. 217, 218
- [113] Marianne Freundlich Smith. The Pontrjagin Duality Theorem in Linear Spaces. *Annals of Mathematics*, 56(2):pp. 248–253, 1952. 145
- [114] Michael B. Smyth. Power Domains and Predicate Transformers: A Topological View. In Josep Diaz, editor, *Automata, Languages and Programming: 10th Colloquium Barcelona, Spain, July 18–22, 1983*, pages 662–675. Springer Berlin Heidelberg, Berlin, Heidelberg, 1983. 3
- [115] Ross Street. The Formal Theory of Monads. *Journal of Pure and Applied Algebra*, 2:149–169, 1972. 34, 66



- [116] Tadeusz Świrszcz. Monadic Functors and Convexity. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Math. Astr. et Phys.*, 22(1):39–42, 1974. 51, 64, 69, 76, 210, 217, 218
- [117] Tadeusz Świrszcz. Monadic Functors and Categories of Convex Sets. Institute of Mathematics of the Polish Academy of Sciences, Preprint 70, 1975. 40
- [118] Masamichi Takesaki. *Theory of Operator Algebra*, volume 1. Springer Verlag, 1979. 105, 143, 205
- [119] Hisaharu Umegaki. Representations and Extremal Properties of Averaging Operators and Their Applications to Information Channels. *Journal of Mathematical Analysis and Applications*, 25:41–73, 1969. 45, 76, 81, 84, 85
- [120] Yemon Choi (<https://mathoverflow.net/users/763/yemon-choi>). Is a  $C^*$ -algebra with an isomorphic predual a von Neumann algebra? MathOverflow. <https://mathoverflow.net/q/244675> (version: 2016-08-04). 105
- [121] Nikolai N. Čencov. *Statistical Decision Rules and Optimal Inference*, volume 53 of *Translations of Mathematical Monographs*. American Mathematical Society, 1982. Translation from the Russian of the 1972 original. 71
- [122] Bram Westerbaan. Quantum Programs as Kleisli Maps. *Electronic Proceedings in Computer Science (EPTCS)*, 236, 2017. 45
- [123] Alexander Wilce. Tensor Products in Generalized Measure Theory. *International Journal of Theoretical Physics*, 31(11):1915–1928, 1992. 4, 88
- [124] Oswald Wyler. Algebraic Theories of Continuous Lattices. In Bernhard Banaschewski and Rudolf-Eberhard Hoffmann, editors, *Continuous Lattices*, volume 871 of *Lecture Notes in Mathematics*, pages 390–413. Springer Berlin Heidelberg, 1981. 226