A Probability Monad on Measure Spaces (Preprint)

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July 29, 2022

Abstract

We define a monad T on a category of measure spaces such that morphisms from 1 to T(X) correspond to probability density functions on X. The Kleisli category of this monad is dual to the category of commutative W*-algebras with normal positive unital maps as morphisms. This is an extension of the probabilistic Gel'fand duality of Bart Jacobs and the author to W*-algebras.

The proof proceeds by showing that the category of W*-algebras is monadic over unital C*-algebras and also over Set, a result of interest in its own right. We then transfer the Radon monad, considered as a comonad on commutative C*-algebras, up to a comonad on commutative W*-algebras (this time with normal unital *-homomorphisms as morphisms), and obtain a monad on compact complete strictly localizable measure spaces by duality.

1 Introduction

Gel'fand duality is the equivalence between the category **CHaus** of compact Hausdorff spaces and $\mathbf{CC^*Alg}^{\mathrm{op}}$, the opposite of the category of commutative unital C*-algebras, with unital *-homomorphisms as morphisms. One direction is given by the functor $C : \mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ that maps a space X to the algebra of complex-valued continuous functions, made into a C*-algebra with pointwise addition and multiplication of functions.

In [14], Bart Jacobs and the author described how to start with the Radon monad $\mathcal{R}$, the natural probability monad on the category **CHaus** of compact Hausdorff spaces, and define a variant of the Gel'fand duality functor to give an equivalence between $\mathcal{K}\ell(\mathcal{R})$ and $\mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$, the category of commutative C*-algebras with positive unital maps (not required to preserve multiplication).

The category of commutative W*-algebras $\mathbf{CW^*Alg}$ is to measure spaces what $\mathbf{CC^*Alg}$ is to compact Hausdorff spaces, with the functor L^∞ playing the role of C. To be specific, we take the category **Meas** to have compact[1] complete strictly localizable measure spaces as objects and equivalence classes of nullset-reflecting measurable maps as morphisms, where the notion of equivalence[2] for measurable maps $f, g : (X, \Sigma_X, \nu_X) \to (Y, \Sigma_Y, \nu_Y)$ is that for all $T \in \Sigma_Y$ we have

[1] This is a measure-theoretic notion, not the topological one, see [10, 342A (c)].

[2] In general this relation is coarser than equality almost everywhere, which would not make L^∞ a faithful functor.

$\nu_X(f^{-1}(T)\triangle g^{-1}(T)) = 0$. Then $L^\infty : \mathbf{Meas} \to \mathbf{CW^*Alg}^{\mathrm{op}}$ is an equivalence, see for instance [24].

A natural question arises as to whether there exists a monad T on **Meas** to play the role of the Radon monad and provide an equivalence $\mathcal{K}\ell(T) \simeq \mathbf{CW^*Alg}_{\mathrm{PU}}$. Since conditional expectations are morphisms in $\mathbf{CW^*Alg}_{\mathrm{PU}}$ this would give a monadic description of conditional expectations.

It turns out that this is the case, although the way of getting it is different. We start out with the observation that the proof of probabilistic Gel'fand duality can be viewed under duality as showing that the inclusion $\mathbf{CC^*Alg} \hookrightarrow \mathbf{CC^*Alg}_{\mathrm{PU}}$ has a left adjoint given by $C\mathcal{S}$, the continuous functions on the state space, and that the coKleisli comparison functor for the comonad also written $C\mathcal{S}$ on $\mathbf{CC^*Alg}$ is an equivalence with $\mathbf{CC^*Alg}_{\mathrm{PU}}$.

The forgetful functors $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$ and $\mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CC^*Alg}_{\mathrm{PU}}$ both have left adjoints, which are essentially the same, given by the *double dual* or *enveloping W*-algebra* $A \mapsto A^{**}$. For a W*-algebra of the form A^{**}, we know what the left adjoint of the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ should be:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) &\cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) \cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B)\\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B).\end{aligned}$$

The trouble is that not every commutative W*-algebra is a double dual. However, the forgetful functor $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$ and its left adjoint -^{**} form a monadic adjunction, and so every commutative W*-algebra is canonically a coequalizer of double duals. This allows us to produce a left adjoint to the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$. The coKleisli comparison functor for the comonad is an equivalence, essentially by the argument given in [34, Theorem 9] (in dual form). We then use the equivalence between $\mathbf{CW^*Alg}^{\mathrm{op}}$ and **Meas** to turn this into a monad T on **Meas** whose Kleisli category is equivalent to ${\mathbf{CW^*Alg}_{\mathrm{PU}}}^{\mathrm{op}}$.

As an explicit calculation, we are able to show that for all finite sets X, made into measure spaces with the counting measure, we have $L^\infty(T(X)) \cong C(2^\omega)^{**}$, and we find a compact complete strictly localizable measure space Y such that $L^\infty(Y) \cong C(2^\omega)^{**}$. If $\nu_d : \mathcal{P}(2^\omega) \to [0,\infty]$ is the counting measure and $\nu_c : \widehat{\mathcal{B}o(2^\omega)} \to [0,1]$ is the completion of the usual probability measure describing an infinite sequence of independent fair coin flips, then

$$Y = (2^\omega, \mathcal{P}(2^\omega), \nu_d) + (2^\omega \times 2^\omega, \mathcal{P}(2^\omega) \otimes \widehat{\mathcal{B}o(2^\omega)}, \nu_d \otimes \nu_c),$$

where $\widehat{\mathcal{B}o(2^\omega)}$ is the completion of the Borel σ-algebra (with respect to ν_c) and $\otimes$ is Fremlin's c.l.d. product [9, Definition 251F].

2 Background on Measure Theory and W*-algebras

2.1 Definitions

We collect various needed facts here with references, as well as establishing the conventions we use when they vary between authors. The general reference texts for C*-algebras and W*-algebras are [5, 32, 17, 26]. If A,B are C*-algebras, a function $f : A \to B$ is a **-homomorphism* if it preserves the $*$ operation and multiplication. A *unital *-homomorphism*. An element of a C*-algebra $a \in A$

is called *positive* iff $a = b^*b$ for some $b \in A$. The set of positive elements form a norm-closed cone $A_+ \subseteq A$ that defines an order on A in the usual way. A linear map $f : A \to B$ is said to be *positive* if $f(A_+) \subseteq B_+$, and this is equivalent to being monotone in terms of the orders defined by the cones. It is easy to prove that a *-homomorphism is positive.

Definition 2.1. *The category* $\mathbf{C^*Alg}_{\mathrm{PU}}$ *has unital C*-algebras as objects and positive unital maps as morphisms, and* $\mathbf{C^*Alg}$ *is the subcategory with the same objects and unital *-homomorphisms as morphisms. The respective full subcategories on commutative C*-algebras are* $\mathbf{CC^*Alg}_{\mathrm{PU}}$ *and* $\mathbf{CC^*Alg}$*. A positive unital map has operator norm* ≤ 1*, so there is a well-defined functor* $\mathrm{Ball} : \mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{Set}$ *that maps a C*-algebra A to its closed unit ball and a positive unital map to its restriction.*

Proof. For the fact that positive unital maps have operator norm ≤ 1, see [29, Theorem 1.3.3]. □

We remark at this point that unlike for Banach algebras and Banach *-algebras there is no need to verify anything about the norm of a *-homomorphism between C*-algebras, so a *-isomorphism is simply a bijective *-homomorphism.

Definition 2.2. *A* W*-algebra *A is a C*-algebra that has a predual A_*,* i.e. *a Banach space such that $A \cong (A_*)^*$ isometrically. The weak-* topology $\sigma(A, A_*)$ is also known as the* ultraweak *or* σ-weak *topology.* □

The above definition is due to Sakai [25], and is a characterization of the C*-algebras that are isomorphic to von Neumann algebras. There are other characterizations, but this one has stood the test of time.

Definition 2.3. *The following are equivalent for a positive linear functional $\phi : A \to \mathbb{C}$, in which case we say ϕ is* normal*:*

(i) ϕ is Scott continuous, i.e. *it preserves suprema of directed joins.*

(ii) ϕ is continuous in the weak- topology $\sigma(A, A_*)$ with respect to a predual A_* of A.*

We say a linear functional ϕ is normal iff it is in the $\mathbb{C}$-linear span of the normal positive linear functionals. This is equivalent to being $\sigma(A, A_*)$ continuous.

Proof. See [26, Theorem 1.13.2]. □

This implies that all preduals define the same weak-* topology, the same set of normal linear functionals, and are all isometrically isomorphic to the set of normal linear functionals via the evaluation embedding $A_* \hookrightarrow A^*$. Therefore we refer to "the predual" of a W*-algebra rather than "a predual", which we can simply take to be the set of normal linear functionals. The uniqueness of preduals does not hold for ordered Banach spaces or order-unit spaces [13, Proposition 3.7, Counterexample 4.1 (i)].

Corollary 2.4. *Let A be a W*-algebra. Every bounded monotone net $(a_i)_{i \in I}$ of self-adjoint elements weakly converges to its least upper bound. If $a \in A$ such that $a_i \to a$ weakly, then a is the least upper bound of $(a_i)_{i \in I}$.*

Proof. First, let a be the least upper bound of $(a_i)_{i \in I}$. For all positive normal linear functionals, $(\phi(a_i))_{i \in I}$ is a monotone net in $\mathbb{R}$ such that $\phi(a) = \sup_{i \in I} \phi(a_i)$, so in $\mathbb{R}$ the number $\phi(a_i) \to \phi(a)$. This implies $a_i \to a$ in $\sigma(A, (A_*)_+)$, where $(A_*)_+$ is the elements of A_* that define positive linear functionals on A. For any vector space, if $X \subseteq A^*$ then $\sigma(A, X) = \sigma(A, \mathrm{span}(X))$, so $a_i \to a$ in $\sigma(A, A_*)$.

Now, since $(a_i)_{i \in I}$ converges weakly to its least upper bound and the weak-* topology is Hausdorff, if $a_i \to a$ then a must be the least upper bound of $(a_i)_{i \in I}$. $\square$

In fact this is true of any dual order-unit space [13, Lemma 2.1 (ii)],[3] and the reader might prefer to prove it directly rather than deduce it from the previous two facts.

Definition 2.5. *The following are equivalent for a positive map $f : A \to B$ between W*-algebras. If one, hence all, of them holds, we say f is* normal.

(i) f is Scott continuous, i.e. *it preserves suprema of directed joins.*

(ii) If ψ is a normal state on B, then $\psi \circ f \in A_*$.

(iii) If $\psi \in B_*$, then $\psi \circ f \in A_*$.

(iv) There is a "pre-adjoint" linear mapping $f_* : B_* \to A_*$ such that for all $a \in A$ and $\psi \in B_*$:

$$\langle f(a), \psi \rangle = \langle a, f_*(\psi) \rangle.$$

(v) f is continuous from $\sigma(A, A_*)$ to $\sigma(B, B_*)$.

Proof. Since the composite of Scott continuous functions is Scott continuous, (i) implies (ii) by Definition 2.3.

Now suppose (ii) holds and $\psi \in B_*$. By taking positive and negative parts of real and imaginary parts, it can be expressed as $\psi_+ - \psi_- + i\psi_{i+} - i\psi_{i-}$ where $\psi_+, \psi_-, \psi_{i+}$ and ψ_{i-} are all positive normal linear functionals[4], which are therefore Scott continuous. So $\psi_s \circ f$ is Scott continuous, and therefore normal, for all $s \in \{+, -, i+, i-\}$, and so $\psi \circ f$ is normal. Therefore (ii) implies (iii).

If (iii) holds, then the map $\text{-} \circ f : B_* \to A_*$ is a pre-adjoint to f:

$$\langle f(a), \psi \rangle = \psi(f(a)) = (\psi \circ f)(a) = \langle a, (\text{-} \circ f)(\psi) \rangle,$$

so (iii) implies (iv).

Conditions (iv) and (v) are equivalent by a standard characterization of weak continuity [27, IV.2.1].

Finally, supposing (v) holds, let $(a_i)_{i \in I}$ be a bounded monotone net in A with supremum a. Then $a_i \to a$ weakly, by Corollary 2.4. Since f is positive, $(f(a_i))_{i \in I}$ a monotone net in B, and it is bounded above by $f(a)$, so converges

[3] An order-unit space is isometrically a dual space iff it is PU-isomorphic to the dual space of a base-norm space, in which case we say it is a dual order-unit space.

[4] See [5, Theorem 12.3.3] for the decomposition into positive and negative parts of a hermitian normal linear functional. The decomposition of a normal linear functional into hermitian real and imaginary parts is more elementary and only really needs the fact that -* is $\sigma(A, A_*)$-continuous [26, Theorem 1.7.8].

weakly to its supremum. By (v), $f(a_i) \to f(a)$, so $f(a)$ is the supremum of of $(f(a_i))_{i\in I}$ (Corollary 2.4 again). Since this holds for all monotone nets, we have proved (i). $\square$

Corollary 2.6. *If A is a W*-algebra and B a C*-algebra and $f : A \to B$ is a *-isomorphism, then B is a W*-algebra and f a* normal *isomorphism.*

Proof. Given a predual A_* and an isometry $i : (A_*)^* \xrightarrow{\sim} A$ we have an isometry $f \circ i : (A_*)^* \xrightarrow{\sim} B$ so B is a W*-algebra. As f is a *-isomorphism, it is a poset isomorphism and therefore preserves directed suprema, so is normal. $\square$

There is also a weaker notion of normality that has some use in relation to measure theory and the dominated convergence theorem.

Definition 2.7. *A positive map $f : A \to B$ between C*-algebras is said to be σ-*normal *iff it preserves suprema of increasing sequences,* i.e. *for each monotone increasing sequence $(a_i)_{i\in\mathbb{N}}$ such that $a = \bigvee_{i\in\mathbb{N}} a_i$ exists, we have that $f(a) = \bigvee_{i\in\mathbb{N}} f(a_i)$. By preservation of $-$, it follows that f also preserves infima of decreasing sequences.* $\square$

Definition 2.8. *We define the following categories with W*-algebras as objects, and the maps as specified:*

(i) Normal positive unital maps: $\mathbf{W^*Alg}_{\mathrm{PU}}$

*(ii) Normal unital *-homomorphisms:* $\mathbf{W^*Alg}$

For the full subcategories of on commutative W-algebras we write a C first:* $\mathbf{CW^*Alg}_{\mathrm{PU}}$ *or* $\mathbf{CW^*Alg}$.

We write U for any of the forgetful functors to the corresponding category of unital C-algebras,* e.g. $\mathbf{W^*Alg}_{\mathrm{PU}} \to \mathbf{C^*Alg}_{\mathrm{PU}}$. *These functors are faithful but neither full nor essentially surjective.* $\square$

In W*-algebras, the *-algebra operations have a useful continuity property.

Theorem 2.9. *If A is a W*-algebra, for each $a \in A$ the operations $a \cdot$ - and - $\cdot\, a : A \to A$ are $\sigma(A, A_*)$-continuous. Similarly, the star operation -* $: A \to A$ is $\sigma(A, A_*)$-continuous.*

Proof. See [26, Theorem 1.7.8]. $\square$

2.2 Products and Quotients of W*-algebras

Definition 2.10. *Let $(B_i)_{i\in I}$ be a family of unital C*-algebras. Define the space $B = \prod_{i\in I} B_i$ to consist of uniformly bounded families $(B_i)_{i\in I}$,* i.e.

$$\prod_{i\in I} B_i = \{(b_i)_{i\in I} \mid b_i \in B_i \text{ and } \exists\alpha \in \mathbb{R}_{\geq 0}.\forall i \in I.\|b_i\|_{B_i} \leq \alpha\}.$$

Make this into a C-algebra by doing all the operations pointwise, and defining the norm as*

$$\|(b_i)_{i\in I}\|_{\prod_{i\in I} B_i} = \sup_{i\in I} \|b_i\|_{B_i}.$$

This is a unital C-algebra with unit* $1_B = (1_{B_i})_{i\in I}$. *Define the maps* $\pi_i : \prod_{i\in I} B_i \to B_i$ *by*

$$\pi_i((b_j)_{j\in I}) = b_i.$$

*These are unital *-homomorphisms. If* A *is a unital C*-algebra and* $(f_i)_{i\in I}$ *a family of positive unital maps with* $f_i : A \to B_i$, *then define* $\langle f_i\rangle_{i\in I} : A \to B_i$ *by*

$$\langle f_i\rangle_{i\in I}(d) = (f_i(d))_{i\in I}.$$

This is the unique positive unital map such that for all $i \in I$, $\pi_i \circ \langle f_j\rangle_{j\in I} = f_i$, *which is a *-homomorphism if all the* f_i *are *-homomorphisms, and defines the product in* $\mathbf{C^*Alg}_{\mathrm{PU}}$ *and* $\mathbf{C^*Alg}$. *If all the* B_i *are W*-algebras, then so is* B *and this defines the product in* $\mathbf{W^*Alg}_{\mathrm{PU}}$ *and* $\mathbf{W^*Alg}$. *If all the* B_i *are commutative, so is* B *and this defines the product in the corresponding full subcategories of commutative C*-algebras and commutative W*-algebras.*

Proof. By Definition A.1, we know that $B = \prod_{i\in I} B_i$ is a Banach space under the given norm. We need to show that the multiplication, involution and unit make it into a C*-algebra. It is clear that the involution is well-defined because the individual involutions preserve the norm. To show that the multiplication is well-defined, regard $\|\text{-}\|_B$ as a function from the set-theoretic product of $(B_i)_{i\in I}$ to $[0,\infty]$, and observe that for $(a_i)_{i\in I}, (b_i)_{i\in I} \in B$

$$\begin{aligned}\|(a_i)_{i\in I}(b_i)_{i\in I}\|_B &= \sup_{i\in I}\|a_ib_i\|_{B_i} \leq \sup_{i\in I}\|a_i\|_{B_i}\|b_i\|_{B_i}\\ &\leq \left(\sup_{i\in I}\|a_i\|_{B_i}\right)\left(\sup_{i\in I}\|b_i\|_{B_i}\right) = \|(a_i)_{i\in I}\|_B\|(b_i)_{i\in I}\|_B.\end{aligned}$$

It is then a simple matter to show that the product is bilinear, and therefore B is a Banach algebra. It can then be verified that -^* makes it into an involutive Banach algebra and 1_B is the unit element. The C*-identity follows from an argument similar to the one above and the fact that $\sup_{i\in I}\alpha_i^2 = (\sup_{i\in I}\alpha_i)^2$ for any family of nonnegative reals $(\alpha_i)_{i\in I}$. It is also easy to verify that for all $i \in I$, the map π_i is a unital *-homomorphism.

In the case that all B_i are W*-algebras, by Theorem A.6 the ℓ^1-direct sum of the preduals forms a predual of the product, and each π_i is normal, by Definition 2.5 (iv). It is direct from the definitions that whether or not all B_i are W*-algebras, the product on B is commutative iff for all $i \in I$, B_i is commutative.

At this point we observe that an element $(b_i)_{i\in I}$ of B is positive iff for all $i \in I$, b_i is positive in B_i. Clearly if $(b_i)_{i\in I}$ is positive, then all the b_i are positive because the π_i are all *-homomorphisms. If each b_i is positive, then there exists some a_i such that $a_i^*a_i = b_i$. Putting them all together, we have $(a_i)_{i\in I}^*(a_i)_{i\in I} = (b_i)_{i\in I}$, so $(b_i)_{i\in I}$ is positive.

Now, let A be a unital C*-algebra and $(f_i)_{i\in I}$ a family of positive unital maps such that $f_i : A \to B_i$. Define $f = \langle f_i\rangle_{i\in I}$ as in Definition A.1. Since $\|f_i\| \leq 1$ for all $i \in I$, as for any positive unital map between C*-algebras, for all $a \in A$:

$$\|\langle f_i\rangle_{i\in I}(a)\|_B = \sup_{i\in I}\|f_i(a)\|_{B_i} \leq \sup_{i\in I}\|a\| = \|a\|,$$

So $\langle f_i\rangle_{i\in I}$ defines a function $A \to \prod_{i\in I} B_i$. It is easy to prove this is linear. If a is positive, then each $f_i(a)$ is positive so $f(a)$ is positive. It is then easy to prove that f is unital if each f_i is unital and to verify the universal property for a product in $\mathbf{C^*Alg}_{\mathrm{PU}}$. It is also easy to verify that if the f_i are all *-homomorphisms, then f is a *-homomorphism, and satisfies the universal property for a product in $\mathbf{C^*Alg}$.

If A and all the B_i are W*-algebras and all the f_i are normal positive unital maps, we can prove that f is normal using Definition 2.5 (i). Let $(a_j)_{j\in J}$ be a bounded monotone net in A with supremum a. We prove $f(a) = \sup_{j\in J} f(a_j)$ as follows. Since f is positive, $f(a_j) \leq f(a)$ for all $j \in J$. To show that $f(a)$ is a *least* upper bound, let $(b_i)_{i\in I} \in B$ such that $f(a_j) \leq (b_i)_{i\in I}$ for all $j \in J$. Then for all $i \in I$, $f_i(a_j) \leq b_i$, so $\sup_{j\in J} f_i(a_j) \leq b_i$, and since f_i is normal, $f_i(a) \leq b_i$. So we have shown that $f(a) \leq (b_i)_{i\in I}$, and therefore $f(a) = \sup_{j\in J} f(a_j)$. The universal property for products in $\mathbf{W^*Alg}_{\mathrm{PU}}$ and $\mathbf{W^*Alg}$ follows.

The commutative cases follow because fully faithful functors reflect limits, and the earlier observation that the product of commutative C*-algebras is commutative. □

Definition 2.11. *Let A be a C*-algebra. A* central projection *is a projection $p \in A$ such that $pa = ap$ for all $a \in A$. Given a central projection p, the* corner defined by p *is pAp, which can equivalently be defined as:*

$$pAp = \{a \in A \mid pap = a\}.$$

This is a C-subalgebra of A with unit p, and the mapping $\pi_p : A \to pAp$ defined by $\pi_p(a) = pap$ is a unital *-homomorphism. If A is a W*-algebra, then pAp is a W*-subalgebra, and π_p is normal.*

Proof. If $a \in pAp$, then there exists $a' \in A$ such that $a = pa'p$. We then have $pap = ppa'pp = pa'p = a$. In the other direction, if $pap = a$, then clearly $a \in pAp$, so the two definitions of the corner defined by p are equivalent, so we will pass back and forth between them from now on without any fanfare.

We first show that pAp it is a *-subalgebra of A. If $a = pap$ and $b = pbp$, then $pa = a$ and $bp = b$, so $pabp = ab$, so pAp is closed under multiplication. Since p is self-adjoint, it is immediate that pAp is closed under $-^*$, and it is a linear subspace of A by bilinearity of multiplication. The unit element is p because $pap = a$ implies $pa = ap = a$. We put off proving that it is a C*-subalgebra for the moment.

In passing, we have proved that π_p is a unital *-homomorphism. It is therefore a unital *-homomorphism considered as a map $\pi_p : A \to A$, so is bounded. If A is a W*-algebra, it is normal because multiplication is separately weak-* continuous in W*-algebras. So the linear map $\pi_p - \mathrm{id}_A : A \to A$ is bounded, and weak-* continuous if A is a W*-algebra. It follows that $(\pi_p - \mathrm{id}_A)^{-1}(0)$ is norm-closed, and weak-* closed if A is a W*-algebra. Since $a \in (\pi_p - \mathrm{id}_A)^{-1}(0)$ iff $pap = a$ iff $a \in pAp$, this shows that pAp is a C*-subalgebra, and a W*-subalgebra if A is a W*-algebra. □

We can now show how to use central projections to describe W*-algebras as products.

Theorem 2.12. *Let A be a W*-algebra, and $(p_i)_{i\in I}$ a family of pairwise orthogonal central projections such that $1 = \bigvee_{i\in I} p_i$. Then $\langle \pi_{p_i} \rangle_{i\in I} : A \to \prod_{i\in I} p_i A p_i$ is a *-isomorphism.*

Proof. It suffices to show that $\langle \pi_{p_i} \rangle_{i \in I}$ is injective and surjective, because then it is an isomorphism of C*-algebras, and therefore a normal isomorphism of W*-algebras by Corollary 2.6.

To prove injectivity, let $a \in A$ such that $\langle \pi_{p_i} \rangle_{i \in I}(a) = 0$. Then for all $i \in I$, $p_i a p_i = 0$, so $p_i a = 0$. Since $(p_i)_{i \in I}$ consists of pairwise orthogonal projections, for each finite $J \Subset I$, $\sum_{i \in J} p_i = \bigvee_{i \in J} p_i$, and so $1 = \sum_{i \in I} p_i$ in the weak-* topology. Therefore

$$0 = \sum_{i \in I} p_i a = \left(\sum_{i \in I} p_i \right) a = a,$$

by separate weak-* continuity of multiplication.

To prove that $\langle \pi_{p_i} \rangle_{i \in I}$ is surjective, we show that its image contains every positive element of $\prod_{i \in I} p_i A p_i$. This suffices, because every element of a C*-algebra is a linear combination of four positive elements. So let $(a_i)_{i \in I} \in \prod_{i \in I} p_i A p_i$ be positive, so for each $i \in I$, a_i is positive in $p_i A p_i$, and therefore also in A, and $p_i a_i = a_i$.

For reasons that will become clear later, we do this for the case of a finite I first. In this case, given $(a_i)_{i \in I}$ in $\prod_{i \in I} p_i A p_i$, we define $a = \sum_{i \in I} a_i$. Then

$$\pi_{p_i}(a) = p_i a = p_i \left(\sum_{j \in I} a_j \right) = p_i \left(\sum_{j \in I} p_j a_j \right) = \sum_{j \in I} p_i p_j a_j = p_i a_i = a_i,$$

so $\langle \pi_{p_i} \rangle_{i \in I}(a) = (a_i)_{i \in I}$. So this proves that in the case that I is finite, $\langle \pi_{p_i} \rangle_{i \in I}$ is a *-isomorphism, and therefore an isometry between the underlying Banach spaces. So for a finite index set I,

$$\left\| \sum_{i \in I} a_i \right\|_A = \|(a_i)_{i \in I}\|_{\prod_{i \in I} p_i A p_i} = \sup_{i \in I} \|a_i\|_A.$$

We now turn to the general case where I is infinite. Define $\left(\sum_{i \in J} a_i \right)_{J \in \mathcal{P}_{\mathrm{fin}}(I)}$. This is a monotone net in A because all the a_i are positive. We show that it is bounded as follows. By the definition of the product, there is some $\alpha \in \mathbb{R}_{\geq 0}$ such that for all $i \in I$, $\|a_i\| \leq \alpha$. For each finite set $J \in \mathcal{P}_{\mathrm{fin}}(I)$, we can define $p_J = \sum_{i \in J} p_i$, which is a projection by pairwise orthogonality. Then we apply the finite case from the previous paragraph to $p_J A p_J$, so we have

$$\left\| \sum_{i \in J} a_i \right\|_A = \left\| \sum_{i \in J} a_i \right\|_{p_J A p_J} = \sup_{i \in J} \|a_i\|_A \leq \alpha.$$

Therefore $\left(\sum_{i \in J} a_i \right)_{J \in \mathcal{P}_{\mathrm{fin}}(I)}$ is a bounded monotone net, so it has a supremum $a \in A$ to which it converges in the weak-* topology. This implies that $a = \sum_{i \in I} a_i$, convergence of the sum being in the weak-* topology. By separate weak-* continuity of multiplication, for all $i \in I$:

$$\pi_{p_i}(a) = p_i a = p_i \left(\lim_{J \in \mathcal{P}_{\mathrm{fin}}(I)} \sum_{j \in J} a_j \right) = \lim_{J \in \mathcal{P}_{\mathrm{fin}}(I)} \sum_{j \in J} p_i p_j a_j = a_i,$$

so $\langle \pi_{p_i} \rangle_{i \in I}(a) = (a_i)_{i \in I}$. $\square$

It is convenient at this point to recall a factorization theorem for *-homomorphisms that uses central projections. We require some facts about about quotients and ideals first.

Proposition 2.13. *Let A be a C*-algebra and $I \subseteq A$ a closed two-sided ideal. Then I is a *-ideal and A/I, equipped with the quotient norm (Definition A.4) is a C*-algebra, and the quotient mapping $q : A \to A/I$ is a *-homomorphism. If A is unital, then A/I is unital with unit $q(1)$.*

Proof. For all but the last sentence, see [5, Proposition 1.8.2] or [32, I.8 Theorem 8.1]. If A is unital, then for all $q(a) \in A/I$ we have $q(a)q(1) = q(a1) = q(a)$ and likewise on the other side, so $q(1)$ is a unit for A/I. □

Proposition 2.14. *Let $f : A \to B$ be a *-homomorphism between C*-algebras. Define*

$$\ker(f) = f^{-1}(0) = \{a \in A \mid f(a) = 0\}$$

*This is a norm-closed *-ideal in A. For each closed two-sided ideal $I \subseteq A$ such that $I \subseteq \ker(f)$, we can define $\tilde{f} : A/I \to B$ as in Definition A.4, which is a *-homomorphism, and the unique function such that $\tilde{f} \circ q = f$, where $q : A \to A/I$ is the quotient mapping.*

Proof. Since *-homomorphisms are contractions, and therefore continuous, $\ker(f)$ is closed, and it is easy to verify that it is a *-ideal. The definition of $\tilde{f}$ as a linear map follows from Definition A.4 directly. We have

$$\tilde{f}([a_1][a_2]) = \tilde{f}([a_1a_2]) = f(a_1a_2) = f(a_1)f(a_2) = \tilde{f}([a_1])\tilde{f}([a_2]),$$

so it is an algebra homomorphism, and

$$\tilde{f}([a]^*) = \tilde{f}([a^*]) = f(a^*) = f(a)^* = \tilde{f}([a])^*,$$

so it is a *-homomorphism, as required. □

Lemma 2.15. *Let A be a W*-algebra and $I \subseteq A$ a subset. The following are equivalent:*

(i) I is a weak- closed two-sided ideal.*

(ii) I is a weak- closed *-ideal.*

(iii) $I = Ap$ for a central projection $p \in A$.

The central projection p is uniquely determined by I, and $I = pA = pAp$. As an algebra I is non-unitally[5] a W-subalgebra of A with unit p.*

Proof. A *-ideal is necessarily two-sided, so (ii) implies (i). For (i) $\Leftrightarrow$ (iii), see [26, Proposition 1.10.5]. If (iii) holds, then since p is central, $Ap = pA = pAp$, and $(Ap)^* = pA = Ap$, so I is a *-ideal, which shows (iii) implies (ii). In the course of the proof in [26, Proposition 1.10.5], the projection p is defined to be the identity element of the W*-subalgebra $I \cap I^*$, and since $I^* = I$ by (ii), this W*-algebra equals I itself. □

[5]It has a unit, but the inclusion mapping is not necessarily a unital *-homomorphism, only a *-homomorphism.

Proposition 2.16. *Let A be a W*-algebra and $I \subseteq A$ a closed two-sided ideal (equivalently *-ideal), which is of the form $I = pAp = pA$ for a unique central projection p (Lemma 2.15).*

*Take $q : A \to A/I$ to be the quotient mapping. This is a normal unital *-homomorphism and $q|_{p^\perp Ap^\perp} : p^\perp Ap^\perp \to A/I$ is a *-isomorphism of C*-algebras, and therefore of W*-algebras.*

*If $f : A \to B$ is a *-homomorphism of C*-algebras such that $I \subseteq \ker(f)$, then under the isomorphism described above, $\tilde{f}$ corresponds to $f|_{p^\perp Ap^\perp}$.*

Proof. We know that A/I is a unital C*-algebra in the quotient norm by Proposition 2.13, and that q is a unital *-homomorphism. We show that $q|_{p^\perp Ap^\perp}$ is bijective, and therefore is a *-isomorphism. Given $[a] \in A/I$, we have $ap \in I$ because $p \in I$. Since p is a central projection, so is $p^\perp$, so

$$p^\perp Ap^\perp \ni p^\perp ap^\perp = ap^\perp = a(1-p) = a - ap.$$

Therefore $q(p^\perp ap^\perp) = [p^\perp ap^\perp] = [a]$, so $q|_{p^\perp Ap^\perp}$ is surjective.

Now suppose that $a_1, a_2 \in p^\perp Ap^\perp$ and $q(a_1) = q(a_2)$. It follows that $a_1 - a_2 \in I$, so $a_1 - a_2 = p(a_1 - a_2)$. We can therefore deduce that

$$a_1 - a_2 = p^\perp a_1 - p^\perp a_2 = p^\perp(a_1 - a_2) = p^\perp p(a_1 - a_2) = 0$$

So $a_1 = a_2$, finishing the proof that $q|_{p^\perp Ap^\perp}$ is a *-isomorphism. Since $p^\perp Ap^\perp$ is a W*-algebra (see Definition 2.11), so is A/I. The quotient map can then be viewed as a composition of the normal map $\pi_{p^\perp}$ from Definition 2.11 and the inverse of $q|_{p^\perp Ap^\perp}$, so q is normal. This can also be proved by taking the subspace of A_* that vanishes on I as a predual of A/I and proving weak-* continuity of q directly.

We have $\tilde{f} \circ q = f$, so $\tilde{f} \circ q|_{p^\perp Ap^\perp} = f|_{p^\perp Ap^\perp}$ as well. ∎

Proposition 2.17. *Let $f : A \to B$ be a normal *-homomorphism of W*-algebras. It factorizes as*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \\ A/\ker(f) \arrow[r, "\tilde{f}"'] & f(A) \arrow[u, "i"'] \end{tikzcd}$$

The ideal $\ker(f) = p_f A p_f$ *where p_f is a central projection which is the largest projection in* $\ker(f)$*, which we call the* kernel projection*. The map q is the quotient mapping, $\tilde{f}$ is as defined in Proposition 2.14 and is an isomorphism, and i is the inclusion mapping. All the objects are W*-algebras, all the morphisms are normal *-homomorphisms, q and $\tilde{f}$ are unital and i is unital iff f is.*

Proof. Since f is normal, the kernel $\ker(f) = f^{-1}(0)$ is weak-* closed. So by Lemma 2.15 it is of the form Ap, or equivalently pAp, for a unique central projection p. It follows that $p \in \ker(f)$, because A is unital. For each projection $p' \in \ker(f) = Ap$, we have $p'p = p'$, so $p' \leq p$, and therefore p is p_f, the join of all projections in $\ker(f)$, which is therefore the largest projection in $\ker(f)$.

The map $q : A \to A/\ker(f)$ restricts to a *-isomorphism $q|_{p^\perp Ap^\perp} p^\perp Ap^\perp \to A/\ker(f)$, so $A/\ker(f)$ is a W*-algebra, and also q is a normal *-homomorphism, by Proposition 2.16, and unital by Proposition 2.13.

The *-homomorphism $\tilde{f}$ is as defined in Proposition 2.14 and makes the whole diagram commute. It is easy to see that it is injective and surjective onto $f(A)$. An injective *-homomorphism between C*-algebras is an isometry onto its image [32, Chapter I, Corollary 5.4], so $f(A)$ is norm-closed in B, so a C*-subalgebra of B, and $\tilde{f}$ is therefore a *-isomorphism, and therefore unital. This is enough to conclude that $f(A)$ is a W*-algebra, but we also need it to be a W*-subalgebra of B, *i.e.* for i to be normal.

By the Banach-Alaoglu theorem, the unit ball of A is weak-* compact, so $f(\mathrm{Ball}(A))$ is weak-* compact in B. We show that $f(\mathrm{Ball}(A)) = \tilde{f}(\mathrm{Ball}(A/\ker(f)))$. If $a \in \mathrm{Ball}(A)$, then $q(a) \in \mathrm{Ball}(A/\ker(f))$ and $\tilde{f}(q(a)) = f(a)$, so $f(\mathrm{Ball}(A)) \subseteq \tilde{f}(\mathrm{Ball}(A/\ker(f)))$. If $b \in \mathrm{Ball}(A/\ker(f))$, then under the *-isomorphism $q|_{p^\perp A p^\perp}$, there exists $a \in \mathrm{Ball}(p^\perp A p^\perp) \subseteq \mathrm{Ball}(A)$ such that $q(a) = b$, and therefore $f(a) = \tilde{f}(b)$, so $\tilde{f}(\mathrm{Ball}(A/\ker(f))) \subseteq f(\mathrm{Ball}(A))$. Since $\tilde{f}$ is an isometry, $\mathrm{Ball}(f(A)) = \tilde{f}(\mathrm{Ball}(A/\ker(f))) = f(\mathrm{Ball}(A))$, which is therefore weak-* compact in B. It follows from the Krein-Šmulian theorem [27, IV.6.4 Corollary] that $f(A)$ is weak-* closed and therefore a W*-subalgebra of B. It follows that $\tilde{f}$ is normal both as a map $A/\ker(f) \to f(A)$ and as a map $A/\ker(f) \to B$, and i is normal.

Lastly, we have that the unit of $f(A)$ is $f(1)$, so i is unital iff f is. □

2.3 Probabilistic Gel'fand Duality for C*-algebras

We summarize the main result of [14] and its dual version that occurs entirely on commutative C*-algebras. We write **CHaus** for the category of compact Hausdorff spaces and continuous maps.

Definition 2.18. *A* state *on a unital C*-algebra A is a positive unital map to $\mathbb{C}$. The set of states is equipped with the weak-* topology $\sigma(A^*, A)$, in which it is a compact Hausdorff space, and this is called the* state space. *We define the functor $\mathcal{S} : \mathbf{C^*Alg}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{CHaus}$ to be the hom functor $\mathbf{C^*Alg}_{\mathrm{PU}}(\text{-}, \mathbb{C})$, but with the state spaces given the weak-* topology.* □

Definition 2.19. *For a compact Hausdorff space X, $C(X)$ is the set of continuous functions to $\mathbb{C}$, made into a commutative unital C*-algebra with pointwise operations and the supremum norm. This defines a functor $C : \mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ if for $f \in \mathbf{CHaus}(X, Y)$ and $b \in C(Y)$ we define $C(f)(b) = b \circ f$.*

For a commutative C-algebra A, the* spectrum $\mathrm{Spec}(A)$ *is the set of unital *-homomorphisms to $\mathbb{C}$. It forms a closed subset of $\mathcal{S}(A)$, and is a functor* $\mathrm{Spec} : \mathbf{CC^*Alg}^{\mathrm{op}} \to \mathbf{CHaus}$, *defined on maps as for $\mathcal{S}$.*

These functors form an adjoint equivalence making $\mathbf{CC^*Alg}^{\mathrm{op}} \simeq \mathbf{CHaus}$, which is known as Gel'fand duality. *The unit and counit are as follows:*

$$\begin{aligned} \eta_X &: X \to \mathrm{Spec}(C(X)) \\ \eta_X(x)(a) &= a(x) \\ \epsilon_A &: A \to C(\mathrm{Spec}(A)) \\ \epsilon_A(a)(\phi) &= \phi(a). \end{aligned}$$

□

Definition 2.20. *Let $\mathcal{R} = \mathcal{S} \circ C$ as a functor. If we take η to be the map with that name in Definition 2.19 and define $\zeta_X : C(X) \to C(\mathcal{R}(X))$ and $\mu_X : \mathcal{R}(\mathcal{R}(X)) \to \mathcal{R}$ by*

$$\zeta_X(a)(\phi) = \phi(a)$$
$$\mu_X(\Phi)(a) = \Phi(\zeta_X(a)).$$

Then $(\mathcal{R}, \eta, \mu)$ is a monad. Define a functor $C_{PU} : \mathcal{K}\ell(\mathcal{R}) \to \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$ on objects as $C_{PU}(X) = C(X)$ and on maps $f : X \to \mathcal{R}(Y)$ by

$$C_{PU}(f) : C(Y) \to C(X)$$
$$C_{PU}(f)(b)(x) = f(x)(b).$$

Then C_{PU} is an equivalence of categories [14, Theorem 5.1]. □

By combining this with Gel'fand duality we get the following.

$$\begin{tikzcd} \mathcal{K}\ell(\mathcal{R}) \arrow[r, "C_{PU}"] \arrow[d, shift left, "G_{\mathcal{R}}"] \arrow[d, phantom, "\dashv"] & \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}} \arrow[d, shift left, "C \circ \mathcal{S}"] \arrow[d, phantom, "\dashv"] \\ \mathbf{CHaus} \arrow[u, shift left, "F_{\mathcal{R}}"] \arrow[r, "C"'] & \mathbf{CC^*Alg}^{\mathrm{op}}, \arrow[u, hook, shift left] \end{tikzcd} \tag{2.21}$$

and this is a morphism of adjunctions in the sense of [20, IV.7]. Removing the opposites on the right hand side, the operator-algebraic adjunction can be expressed as $\mathbf{CC^*Alg}(C(\mathcal{S}(A)), B) \cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B)$.

2.4 The Enveloping W*-algebra of a C*-algebra

The forgetful functors U from Definition 2.8 have left adjoints given on objects by the *enveloping W*-algebra*, a construction introduced by Sherman [28] and Takeda [31]. The universal property for *-homomorphisms is described by Dauns in [4, §3.1]. We need the universal property for positive unital maps, so we go over the proof of this as well, while explaining some notations and clearing up a point that sometimes causes confusion.

The underlying Banach space of the enveloping W*-algebra is A^{**}. It is helpful to have an explicit description of the product and involution to work with, although if one knows that A^{**} is supposed to be a W*-algebra and the evaluation mapping $\mathrm{ev} : A \to A^{**}$ a *-homomorphism then they could be defined "by continuity". We find this approach unsatisfying and there is a vexing subtlety to it that I will point out in Warning 2.26.

We take the definition of the dual norm for granted, and the fact that if E is a Banach space, the evaluation mapping $\mathrm{ev} : E \to E^{**}$ is isometric onto its image.

Definition 2.22 (Arens Products)**.** *Let A be a Banach algebra. There are two ways to define an associative Banach algebra structure on A^{**}, called the first and second Arens products, shown on the left and right respectively of the third row below. In the following, $a, b \in A$, $\phi \in A^*$ and $\Gamma, \Delta \in A^{**}$:*

$$(\phi \triangleleft a)(b) = \phi(ab) \qquad (a \triangleright \phi)(b) = \phi(ba)$$
$$(\Gamma \triangleleft \phi)(a) = \Gamma(\phi \triangleleft a) \qquad (\phi \triangleright \Delta)(a) = \Delta(a \triangleright \phi)$$
$$(\Gamma \triangleleft \Delta)(\phi) = \Gamma(\Delta \triangleleft \phi) \qquad (\Gamma \triangleright \Delta)(\phi) = \Delta(\phi \triangleright \Gamma)$$

Then $\mathrm{ev} : A \to A^{**}$ *is a Banach algebra homomorphism for both products. In fact the products agree if one element is* $\mathrm{ev}(a)$ *because of a relation between the multiplications on* A^**:*

$$\begin{aligned} \phi \triangleright \mathrm{ev}(a) &= \phi \triangleleft a & \quad \textit{and} \quad & \mathrm{ev}(a) \triangleleft \phi = a \triangleright \phi \\ \mathrm{ev}(a) \triangleleft \Delta &= \mathrm{ev}(a) \triangleright \Delta & \quad \textit{and} \quad & \Gamma \triangleleft \mathrm{ev}(a) = \Gamma \triangleright \mathrm{ev}(a). \end{aligned}$$

If A *has a unit* 1*, the element* $\mathbb{1} = \mathrm{ev}(1) \in A^{**}$ *is a unit for both of the above multiplications, and so* $\mathrm{ev} : A \to A^{**}$ *is unital.*

The algebra A *is said to be* Arens regular *iff* $\Gamma \triangleleft \Delta = \Gamma \triangleright \Delta$ *for all* $\Gamma, \Delta \in A^{**}$*. If* A *is a Banach *-algebra, define a star on* A^{**} *by:*

$$\begin{aligned} \phi^*(a) &= \overline{\phi(a^*)} \\ \Gamma^*(\phi) &= \overline{\Gamma(\phi^*)}. \end{aligned}$$

This is an antilinear involution of norm 1 *on* A^{**}*. We have* $\mathrm{ev}(a^*) = \mathrm{ev}(a)^*$ *and* $(\Gamma \triangleleft \Delta)^* = \Delta^* \triangleright \Gamma^*$*.*

Proof. We give the proof as a sequence of smaller statements to be proved, each of which can be proved by elementary algebraic arguments. We miss out the the steps for proving that $\triangleright$ defines a Banach algebra multiplication on A^{**}, as the reader can write the proof that $\triangleleft$ is one using symmetrical letters, turn it round, hold it up to a mirror, and interpret it according to the convention that arguments precede functions, rather than follow them, in order to get a proof for $\triangleright$.

1. $\phi \triangleleft a : A \to \mathbb{C}$ is linear.
2. $\|\phi \triangleleft a\| \leq \|\phi\| \cdot \|a\|$, so $\triangleleft$ defines a function $A^* \times A \to A^*$:
3. $\phi \triangleleft \text{-} : A \to A^*$ is a linear map:
4. $\text{-} \triangleleft a : A^* \to A^*$ is a linear map:
5. $(\phi \triangleleft a) \triangleleft b = \phi \triangleleft ab$:
6. $\Gamma \triangleleft \phi : A \to \mathbb{C}$ is linear: by 3.
7. $\|\Gamma \triangleleft \phi\| \leq \|\Gamma\| \cdot \|\phi\|$, so $\triangleleft$ defines a function $A^{**} \times A^* \to A^*$: by 2.
8. $\Gamma \triangleleft \text{-} : A^* \to A^*$ is a linear map: by 4.
9. $\text{-} \triangleleft \phi : A^{**} \to A^*$ is a linear map.
10. $\Gamma \triangleleft \Delta : A^* \to \mathbb{C}$ is linear: by 8.
11. $\|\Gamma \triangleleft \Delta\| \leq \|\Gamma\| \cdot \|\Delta\|$, so $\Gamma \triangleleft \Delta \in A^{**}$: by 7.
12. $\Gamma \triangleleft \text{-} : A^{**} \to A^{**}$ is a linear map: by 9.
13. $\text{-} \triangleleft \Delta : A^{**} \to A^{**}$ is a linear map.
14. $(\Gamma \triangleleft \phi) \triangleleft a = \Gamma \triangleleft (\phi \triangleleft a)$: by 5.
15. $(\Gamma \triangleleft \Delta) \triangleleft \phi = \Gamma \triangleleft (\Delta \triangleleft \phi)$: by 14.

16. $(\Gamma \triangleleft \Delta) \triangleleft \Theta = \Gamma \triangleleft (\Delta \triangleleft \Theta)$: by 15.

So we have proved that A^{**} is a Banach algebra under $\triangleleft$.

17. $\mathrm{ev}(a) \triangleleft \mathrm{ev}(b) = \mathrm{ev}(ab)$.

So ev is a Banach algebra homomorphism. We know from Banach space theory that it is an isometry, and therefore injective with closed range.

18. $\phi \triangleright \mathrm{ev}(a) = \phi \triangleleft a$.

19. $\mathrm{ev}(a) \triangleleft \Delta = \mathrm{ev}(a) \triangleright \Delta$: by 18.

The proof that 19 is true on the other side is similar, and omitted. Now let A have a unit 1, and define $\mathbb{1} = \mathrm{ev}(1)$.

20. $\phi \triangleleft 1 = \phi$.

21. $\mathbb{1} \triangleleft \phi = \phi$.

22. $\mathbb{1} \triangleleft \Delta = \Delta$: by 20.

23. $\Gamma \triangleleft \mathbb{1} = \Gamma$: by 21.

So $(A^{**}, \triangleleft, \mathbb{1})$ is a unital Banach algebra and ev is a unital Banach algebra homomorphism. The proof of the corresponding facts for $\triangleright$ is similar, and therefore omitted.

Now, suppose that A is a Banach *-algebra, and define the stars on A^* and A^{**} as in the definition above.

24. $\phi^* : A \to \mathbb{C}$ is linear.

25. $\|\phi^*\| \leq \|\phi\|$, so -^* defines a function $A^* \to A^*$.

26. $\phi^{**} = \phi$, and so $\|\phi^*\| = \|\phi\|$.

27. $\text{-}^* : A^* \to A^*$ is antilinear.

28. $\Gamma^* : A^* \to \mathbb{C}$ is linear: by 27.

29. $\|\Gamma^*\| \leq \|\Gamma\|$, so -^* defines a function $A^{**} \to A^{**}$: by 25.

30. $\Gamma^{**} = \Gamma$, so $\|\Gamma^*\| = \|\Gamma\|$: by 26.

31. $\text{-}^* : A^{**} \to A^{**}$ is antilinear.

32. $\mathrm{ev}(a^*) = \mathrm{ev}(a)^*$.

33. $(\phi \triangleleft a)^* = a^* \triangleright \phi^*$.

34. $(\Gamma \triangleleft \phi)^* = \phi^* \triangleright \Gamma^*$: by 33 and 26.

35. $(\Gamma \triangleleft \Delta)^* = \Delta^* \triangleright \Gamma^*$: by 34 and 30.

$\square$

Lemma 2.23. *Let A be a Banach algebra. For all $\Gamma \in A^{**}$, the mappings $A^{**} \to A^{**}$ defined by $\text{-} \triangleleft \Gamma$ and $\Gamma \triangleright \text{-}$ are $\sigma(A^{**}, A^*)$-continuous. If A is an involutive Banach algebra, then the involution $\text{-}^* : A^{**} \to A^{**}$ is $\sigma(A^{**}, A^*)$-continuous.*

Proof. The mapping $\text{-}\triangleleft\Delta : A^{**} \to A^{**}$ is continuous for the $\sigma(A^{**}, A^*)$ topology because it is the adjoint [27, IV.2.1] of $\Delta \triangleleft \text{-} : A^* \to A^*$ under the pairing $A^{**} \times A^* \to \mathbb{C}$ defined by evaluation:

$$\langle \Gamma \triangleleft \Delta, \phi \rangle = (\Gamma \triangleleft \Delta)(\phi) = \Gamma(\Delta \triangleleft \phi) = \langle \Gamma, \Delta \triangleleft \phi \rangle.$$

The proof that $\Gamma \triangleright \text{-} : A^{**} \to A^{**}$ is $\sigma(A^{**}, A^*)$-continuous is similar.

To prove $\text{-}^* : A^{**} \to A^{**}$ is $\sigma(A^{**}, A^*)$-continuous, we use the same kind of reasoning. We have, taking $\Gamma \in A^{**}$ and $\phi \in A^*$:

$$\overline{\langle (\text{-}^*)(\Gamma), \phi \rangle} = \overline{\Gamma^*(\phi)} = \overline{\overline{\Gamma(\phi^*)}} = \Gamma(\phi^*) = \langle \Gamma, (\text{-}^*)(\phi) \rangle.$$

□

Proposition 2.24. *Let A be a Banach algebra. The following are equivalent:*

(i) A is Arens regular, i.e. *for all $\Gamma, \Delta \in A^{**}$, $\Gamma \triangleleft \Delta = \Gamma \triangleright \Delta$.*

(ii) For all $\Gamma \in A^{**}$, $\Gamma \triangleleft \text{-}$ is $\sigma(A^{**}, A^*)$-continuous.

(iii) For all $\Delta \in A^{**}$, $\text{-} \triangleright \Delta$ is $\sigma(A^{**}, A^*)$-continuous.

*If A is a Banach *-algebra, we can add:*

*(iv) A^{**} is a Banach *-algebra under $\triangleleft$ and -^* in Definition 2.22.*

*(v) A^{**} is a Banach *-algebra under $\triangleright$ and -^* in Definition 2.22.*

*(vi) A^{**} is a Banach *-algebra under $\triangleleft$ or $\triangleright$ under an involution $\text{-}^\star$ that is $\sigma(A^{**}, A^*)$-continuous and for which $\mathrm{ev} : A \to A^{**}$ is a *-homomorphism.*

Proof. It is immediate from Lemma 2.23 that (i) implies (ii) and (iii).

Now, by a standard fact [27, IV.1.3], the image of A in A^{**} under ev is $\sigma(A^{**}, A^*)$-dense, and by Definition 2.22 for all $a \in A$, $\Gamma \triangleleft \mathrm{ev}(a) = \Gamma \triangleright \mathrm{ev}(a)$. Therefore if (ii) holds, for all $\Gamma \in A^{**}$, we have $\Gamma \triangleleft \text{-} = \Gamma \triangleright \text{-}$ by $\sigma(A^{**}, A^*)$-continuity, so (i) holds. The proof that (iii) implies (i) is similar.

If A is a Banach *-algebra, we know from Definition 2.22 that (i) implies (iv), (v). It also follows from Lemma 2.23 that both (iv) and (v) individually imply (vi). We also have that (vi) implies that either (iv) or (v) holds: If $(A^{**}, \triangleleft, \text{-}^\star)$ is a Banach *-algebra satisfying the conditions of (vi), then since the canonical -^* is also $\sigma(A^{**}, A^*)$-continuous and agrees with $\text{-}^\star$ on the dense image of A under ev, so the two are equal by continuity.

To finish the proof, observe that (iv) implies (ii) and (v) implies (iii). For (iv), this is because $\Gamma \triangleleft \text{-} = \text{-}^* \circ (\text{-} \triangleright \Gamma^*)$, a composite of $\sigma(A^{**}, A^*)$-continuous functions. The argument that (v) implies (iii) is similar. □

Proposition 2.25. *If A is commutative, then for all $\Gamma, \Delta \in A^{**}$, $\Gamma \triangleleft \Delta = \Delta \triangleright \Gamma$. Therefore a Banach algebra A is Arens regular and commutative iff A^{**} is commutative under either of the products $\triangleleft$ or $\triangleright$.*

Proof. Assume that A is commutative. By direct algebraic manipulation it is easy to prove that for all $a \in A$, $\phi \in A^*$ and $\Gamma, \Delta \in A^{**}$ that $\phi \triangleleft a = a \triangleleft \phi$, therefore $\Gamma \triangleleft \phi = \phi \triangleright \Gamma$, and therefore $\Gamma \triangleleft \Delta = \Delta \triangleright \Gamma$.

Therefore we have shown that if A is commutative and Arens regular, then A^{**} is commutative in both Arens products.

Now, if A^{**} is commutative in either Arens product, then so is A because ev is an injective algebra homomorphism for both Arens products. Furthermore, without loss of generality assuming that A^{**} is commutative in $\triangleleft$, we have $\Gamma \triangleleft \Delta = \Delta \triangleleft \Gamma = \Gamma \triangleright \Delta$, so A is Arens regular. $\square$

Warning 2.26.

One might think that if a Banach *-algebra A is commutative, then A^{**} is commutative and so A is Arens regular. One might even think that all Banach *-algebras, or at least all used in practice, are Arens regular, by some weak continuity argument or other. Neither of these things is so. The convolution *-algebra $\ell^1(\mathbb{Z})$ is *not* Arens regular [2, Theorem 3.1], so the -* operation on $\ell^1(\mathbb{Z})^{**}$ makes neither $\triangleleft$ nor $\triangleright$ into a Banach *-algebra product and $\ell^1(\mathbb{Z})^{**}$ is not commutative in either product. In fact a locally compact group G is finite iff $L^1(G)$ is Arens regular [35]. **For a C*-algebra A, we show below that A is Arens regular and that A^{**} is commutative iff A is, but it follows from the above discussion that this cannot be proved by any elementary algebraic argument or one involving properties of weak-* topologies of arbitrary dual or double dual Banach spaces.** The textbook [23, p. 47] warns of the "unusual number of false results" published in this area. $\square$

Parts 1 to 5 of the proof of Definition 2.22 can be interpreted as saying $\phi \triangleleft a$ defines a right module structure of A on A^* for any Banach algebra, and the mirrored version is that $a \triangleright \phi$ defines a left module structure. We will need the fact that this module structure interacts nicely with homomorphisms. It is well known in ring theory that a ring homomorphism $f : A \to B$ defines a way of turning a B-module into an A module.

Normally I would write f^* for the following operation. Unfortunately that conflicts with the need to use -* in C*-algebras.

Lemma 2.27. *Let E, F be Banach spaces with preduals E_*, F_* respectively. Let $f : F_* \to E_*$ be a bounded linear map. Then there exists an adjoint map $f^\sigma : E \to F$, which is bounded and continuous from $\sigma(E, E_*)$ to $\sigma(F, F_*)$. In the case that the pairings are the transposed dual pairings, f^σ can be defined for $x \in E$ by $f^\sigma(x) = x \circ f$.*

Proof. First define $g : (E_*)^* \to (F_*)^*$ by $g(x) = x \circ f$. This map is bounded. Let $i_E : E \xrightarrow{\sim} (E_*)^*$ and $i_F : F \xrightarrow{\sim} (F_*)^*$ be the isometries making E_* and F_* preduals of E and F, respectively, and $\langle -,- \rangle_E : E \times E_* \to \mathbb{C}$ and $\langle -,- \rangle_F : F \times F_* \to \mathbb{C}$ the bilinear pairings defined by uncurrying these maps. Define $f^\sigma = i_F^{-1} \circ g \circ i_E$. Since i_E and i_F are isometries, f^σ is bounded. For all $x \in E$ and $\psi \in F_*$:

$$\begin{aligned}\langle f^\sigma(x), \psi \rangle_F &= i_F(f^\sigma(x))(\psi) = g(i_E(x))(\psi) = (i_E(x) \circ f)(\psi) = i_E(x)(f(\psi)) \\ &= \langle x, f(\psi) \rangle_E.\end{aligned}$$

It follows from [27, IV.2.1] that f^σ is continuous from $\sigma(E, E_*)$ to $\sigma(F, F_*)$, because it has an adjoint map. $\square$

Lemma 2.28. *Let $f : A \to B$ be a homomorphism of Banach algebras. Then $f^\sigma : B^* \to A^*$ (recall Lemma 2.27) is a homomorphism of right A-modules with* *respect to $\triangleleft$, and a homomorphism of left A-modules with respect to $\triangleright$,* i.e. *for all $a \in A$, $\psi \in B^*$:*

$$f^\sigma(\psi \triangleleft f(a)) = f^\sigma(\psi) \triangleleft a \qquad\qquad f^\sigma(f(a) \triangleright \psi) = a \triangleright f^\sigma(\psi).$$

Proof. Simple algebraic manipulations using Definition 2.22 and Lemma 2.27. □

We also need a fact about this module structure when the underlying Banach algebra is a W*-algebra, as well as the -^* defined on the dual space.

Lemma 2.29. *Let A be a W*-algebra. For both $\triangleleft$ and $\triangleright$, A_* is a submodule of A^*,* i.e. *for all $a \in A$ and $\phi \in A_*$, $\phi \triangleleft a$ and $a \triangleright \phi$ are in A_*. Furthermore, A_* is invariant under -^*,* i.e. *if $\phi \in A_*$, then $\phi^* \in A_*$.*

Proof. We consider A_* to be the set of $\sigma(A, A_*)$-continuous linear functionals on A. So we want to show that if ϕ is $\sigma(A, A_*)$-continuous, then so are $\phi \triangleleft a$ and $a \triangleright \phi$ for all $a \in A$. Since $(\phi \triangleleft a)(b) = \phi(ab)$, the mapping $\phi \triangleleft a$ is the composite of $b \mapsto ab$ and ϕ. Since multiplication is separately weakly continuous on each side in W*-algebras (Theorem 2.9), it follows that $\phi \triangleleft a$ is $\sigma(A, A_*)$-continuous and hence an element of A_*. The argument for $a \triangleright \phi$ is similar, using separate continuity on the other side.

The -^* map $A \to \overline{A}$ is $\sigma(A, A_*)$-continuous by Theorem 2.9 so for all $\phi \in A_*$, $\overline{\langle \text{-}, \phi\rangle} \circ (\text{-}^*)$ is $\sigma(A, A_*)$-continuous. Given $a \in A$, we see

$$(\overline{\langle \text{-}, \phi\rangle} \circ (\text{-}^*))(a) = \overline{\langle a^*, \phi\rangle} = \overline{\phi(a^*)} = \phi^*(a),$$

so we have shown that ϕ^* is $\sigma(A, A_*)$-continuous and therefore an element of A_*. □

At last, we can define the enveloping W*-algebra.

Definition 2.30. *Let A be a unital C*-algebra. Then A is Arens regular and A^{**} is a C*-algebra, and since it is a dual space, a W*-algebra. It is commutative iff A is. The evaluation mapping $\eta_A = \mathrm{ev} : A \to A^{**}$ is a unital *-homomorphism. If B is a W*-algebra and $f : A \to B$ a positive unital map, then there is a unique normal positive unital map $\tilde{f} : A^{**} \to B$ such that the following commutes:*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & A^{**} \arrow[d, dotted, "\tilde{f}"] \\ & B, \end{tikzcd} \tag{2.31}$$

*and $\tilde{f}$ is a *-homomorphism or unital iff f is, respectively.*

*It follows that -^{**} extends to a functor that is left adjoint to the forgetful functor U with unit η where U any of the four forgetful functors involving positive maps or *-homomorphisms for noncommutative W*-algebras in Definition 2.8 (completely positive maps will be handled later) and any of the four forgetful functors for the full subcategories on commutative W*-algebras.*

On maps, $f : A \to B$:

$$f^{**}(\Gamma)(\psi) = \Gamma(\psi \circ f).$$

By the universal property above, for all these adjunctions the counit $\epsilon_A : A^{**} \to A$ for a W-algebra A is a normal *-homomorphism, and the unique map such that for all $\Gamma \in A^{**}$ and $\phi \in A_*$:*

$$\phi(\epsilon_A(\Gamma)) = \Gamma(\phi).$$

Proof. We will prove the universal property for *-homomorphisms first, and then use it to finally prove that A^{**} is a C*-algebra, and therefore a W*-algebra, and then deal with the other cases. However, we will start with the definition of $\tilde{f}$ and the uniqueness part of the universal property in the most general case, positive unital maps.

Let $f : A \to B$ be positive and unital, and therefore bounded. We have $f^\sigma : B^* \to A^*$ (recall Lemma 2.27) so we can define $g : B_* \to A^*$ to be the restriction of this map to B_*, on which it is still bounded and linear. Now define $\tilde{f} = g^\sigma : A^{**} \to B$. By Lemma 2.27 this is continuous from $\sigma(A^{**}, A^*)$ to $\sigma(B, B_*)$, and for all $\Gamma \in A^{**}$, the element $\tilde{f}(\Gamma) \in B$ is characterized by the property that for all $\psi \in B_*$:

$$\psi(\tilde{f}(\Gamma)) = \Gamma(f^\sigma(\psi)). \tag{2.32}$$

For each $a \in A$, and $\psi \in B_*$ we have

$$\psi(\tilde{f}(\eta_A(a))) = \eta_A(a)(f^\sigma(\psi)) = f^\sigma(\psi)(a) = \psi(f(a)),$$

so $\tilde{f} \circ \eta_A = f$, *i.e.* $\tilde{f}$ makes the diagram (2.31) commute. Since $\eta_A(A) \subseteq A^{**}$ is $\sigma(A^{**}, A^*)$-dense (a standard fact about weak topologies [27, IV.1.3]), for any function $h : A^{**} \to B$ that is continuous from $\sigma(A^{**}, A^*)$ to $\sigma(B, B_*)$ such that $h \circ \eta_A = f$, it must be that $h = \tilde{f}$. This establishes the uniqueness part of the universal property in all cases.

We also have that if f is unital, then since the unit of A^{**} is defined to be $\mathbb{1} = \mathrm{ev}(1) = \eta_A(1)$, then $\tilde{f}$ is unital. Therefore we will concentrate on the sub-unital versions because the unital versions will take care of themselves.

Now, take f to be a *-homomorphism. We first show that $\tilde{f}$ preserves $\triangleleft$, $\triangleright$ and -*.

Let $\Gamma, \Delta \in A^{**}$ and $\psi \in B_*$:

$$\psi(\tilde{f}(\Gamma \triangleleft \Delta)) = (\Gamma \triangleleft \Delta)(f^\sigma(\psi)) = \Gamma(\Delta \triangleleft f^\sigma(\psi)),$$

and

$$\psi(\tilde{f}(\Gamma)\tilde{f}(\Delta)) = (\tilde{f}(\Delta) \triangleright \psi)(\tilde{f}(\Gamma)) = \Gamma(f^\sigma(\tilde{f}(\Delta) \triangleright \psi)),$$

by Lemma 2.29 and (2.32). Then for all $a \in A$:

$$\begin{aligned}
(\Delta \triangleleft f^\sigma(\psi))(a) &= \Delta(f^\sigma(\psi) \triangleleft a) \\
&= \Delta(f^\sigma(\psi \triangleleft f(a))) && \text{by Lemmas 2.28 and 2.29} \\
&= (\psi \triangleleft f(a))(\tilde{f}(\Delta)) \\
&= \psi(f(a)\tilde{f}(\Delta)) \\
&= (\tilde{f}(\Delta) \triangleright \psi)(f(a)) \\
&= f^\sigma(\tilde{f}(\Delta) \triangleright \psi)(a),
\end{aligned}$$

so

$$\psi(\tilde{f}(\Gamma \triangleleft \Delta)) = \Gamma(\Delta \triangleleft f^\sigma(\psi)) = \Gamma(f^\sigma(\tilde{f}(\Delta) \triangleright \psi)) = \psi(\tilde{f}(\Gamma)\tilde{f}(\Delta)),$$

and therefore $\tilde{f}(\Gamma \triangleleft \Delta) = \tilde{f}(\Gamma)\tilde{f}(\Delta)$. The proof that $\tilde{f}(\Gamma \triangleright \Delta) = \tilde{f}(\Gamma)\tilde{f}(\Delta)$ is similar. For the star, observe that given $\Gamma \in A^{**}$ and $\psi \in B_*$, we have

$$\psi(\tilde{f}(\Gamma^*)) = \Gamma^*(f^\sigma(\psi)) = \overline{\Gamma(f^\sigma(\psi)^*)},$$

and by Lemma 2.29,

$$\psi(\tilde{f}(\Gamma)^*) = \overline{\psi^*(\tilde{f}(\Gamma))} = \overline{\Gamma(f^\sigma(\psi^*))}.$$

For all $a \in A$ we have

$$f^\sigma(\psi)^*(a) = \overline{f^\sigma(\psi)(a^*)} = \overline{\psi(f(a^*))} = \overline{\psi(f(a)^*)} = \psi^*(f(a)) = f^\sigma(\psi^*)(a),$$

so

$$\psi(\tilde{f}(\Gamma^*)) = \overline{\Gamma(f^\sigma(\psi)^*)} = \overline{\Gamma(f^\sigma(\psi^*))} = \psi(\tilde{f}(\Gamma)^*),$$

and therefore $\tilde{f}(\Gamma^*) = \tilde{f}(\Gamma)^*$.

So we have almost proved that $\tilde{f}$ is a *-homomorphism. Unfortunately we still have not proved that A^{**} is a C*-algebra and that the two Arens products coincide. Let $(\mathcal{H}_u, \pi_u)$ be the universal representation of A, the direct sum of all GNS representations [32, III Theorem 2.4] so $\mathcal{H}_u = \bigoplus_{\phi \in A^*_+)} \mathcal{H}_\phi$ and $\pi_u : A \to B(\mathcal{H}_u)$ is the direct sum of the GNS representations of each $\phi \in A^*_+$. We have a map $\tilde{\pi_u} : A^{**} \to B(\mathcal{H})$.

First we show that $\tilde{\pi_u}$ is injective. Since linear functionals on A can be expressed as linear combinations of positive linear functionals, for every $\Gamma \in A^{**}$ such that $\Gamma \neq 0$, there is some $\phi \in A^*_+$ such that $\Gamma(\phi) \neq 0$. Let's write ξ_ϕ for the vector in $\mathcal{H}_u$ such that $\langle \xi_\phi, \pi_u(a)(\xi_\phi)\rangle = \phi(a)$ for all $a \in A$, which exists by the definition of a GNS representation [32, I Theorem 9.14]. Then by (2.32) and the normality of the linear functional $\langle \xi_\phi, (\text{-})\xi_\phi\rangle$ we have

$$\langle \xi_\phi \tilde{\pi_u}(\Gamma)(\xi_\phi)\rangle = \Gamma(\pi_u^\sigma(\langle \xi_\phi, (\text{-})(\xi_\phi)\rangle)) = \Gamma(\phi) \neq 0,$$

so $\tilde{\pi_u}(\Gamma) \neq 0$. Therefore $\tilde{\pi_u}$ is injective. So

$$\tilde{\pi}(\Gamma \triangleleft \Delta) = \tilde{\pi}(\Gamma)\tilde{\pi}(\Delta) = \tilde{\pi}(\Gamma \triangleright \Delta)$$

implies that the two products coincide, and therefore A^{**} is a Banach *-algebra, so A is Arens regular and A^{**} is commutative iff A is by Proposition 2.25.

Now we show that the double dual norm in A^{**} is equivalent to the operator norm on $B(\mathcal{H}_u)$ under the mapping $\tilde{\pi_u}$, which finally establishes that the double-dual norm on A^{**} makes it into a C*-algebra.

By Goldstine's theorem [7, V.4.5 Theorem], $\eta_A(\mathrm{Ball}(A))$ is $\sigma(A^{**}, A^*)$-dense in $\mathrm{Ball}(A^{**})$, which is $\sigma(A^{**}, A^*)$-compact by Banach-Alaoğlu, so the image $\tilde{\pi_u}(\mathrm{Ball}(A^{**}))$ is compact and $\tilde{\pi_u}$ is a homeomorphism of $\mathrm{Ball}(A^{**})$ onto its image. By Kaplansky's density theorem (see [6, I.3 §5, Theorem 3] or [32, Chapter II, Theorem 4.8]) the weak closure of $\pi_u(\mathrm{Ball}(A))$ is the unit ball of the weak closure of $\pi_u(A)$. Therefore $\mathrm{Ball}(B(\mathcal{H}_u)) \cap \tilde{\pi_u}(A^{**}) = \tilde{\pi_u}(\mathrm{Ball}(A^{**}))$, so $\tilde{\pi_u}$ preserves norm. Since the operator norm on $B(\mathcal{H}_u)$ is a C*-norm, so is the double-dual norm on A^{**}, so A^{**} is a C*-algebra, and as it is a dual space, a W*-algebra.

Therefore we have proved the universal properties for $\mathbf{W^*Alg}$ and $\mathbf{CW^*Alg}$, so two down and two to go. Since we have established that A^{**} is commutative iff A is, the rest of the proof will only be for the noncommutative case, as the commutative case is implied.

Suppose that $f : A \to B$ is positive and unital. We need to show that $\tilde{f}$ is positive for the positive cone of A^{**} as a C*-algebra. So we need to return briefly to the universal representation to establish the following fact: If $\Gamma \in A^{**}$ is positive in the sense of C*-algebras, then for all $\phi \in A^*_+$, $\Gamma(\phi) \geq 0$. By the definition of the universal representation, there exists ξ_ϕ in $\mathcal{H}_u$ such that $\phi(a) = \langle \xi_\phi, \pi_u(a)(\xi_\phi)\rangle$ for all $a \in A$. Since Γ is positive and $\tilde{\pi_u}$ is a *-homomorphism, $\tilde{\pi_u}(\Gamma)$ is a positive operator on $\mathcal{H}_u$, so:

$$0 \leq \langle \xi_\phi, \tilde{\pi_u}(\Gamma)(\xi_\phi)\rangle = \Gamma(\pi_u^\sigma(\langle \xi_\phi, (\text{-})(\xi_\phi)\rangle)) = \Gamma(\phi),$$

so $\Gamma(\phi) \geq 0$ for all $\phi \in A^*_+$.

With this fact in hand, we can prove that $\tilde{f}$ is positive for $f : A \to B$ a positive bounded map as follows. In a W*-algebra, the positive cone is weak-* closed [26, 1.7.1 Lemma], and it is convex and contains 0 because it is a cone. It then follows by the bipolar theorem [27, IV.1.5 Theorem] that an element $b \in B$ is positive iff $\psi(b) \geq 0$ for all positive $\psi \in B_*$. So to prove that $\tilde{f}$ is positive, it suffices to show that for all $\Gamma \in A^{**}$ and for all positive normal linear functionals $\psi \in B_*$ that $\psi(\tilde{f}(\Gamma)) \geq 0$. By (2.32)

$$\psi(\tilde{f}(\Gamma)) = \Gamma(\psi \circ f) \geq 0,$$

because $\psi \circ f$ is positive in A^*, and we proved in the last paragraph that Γ is positive on positive elements of A^*. This concludes the proof of the universal property for the positive case.

Now we only need to prove the formulas for the functor -^{**} and the counit $\epsilon_A : A^{**} \to A$. By translating from the universal property formulation of an adjunction to the functorial form [20, IV.1 Theorem 2], we know that for a map $f : A \to B$ between C*-algebras, $f^{**} = \widetilde{\eta_B \circ f}$. To evaluate this, we will use the fact that $\mathrm{ev} : B^* \to B_*^{**}$ is an isomorphism, by uniqueness of preduals. Let $\Gamma \in A^{**}$ and $\psi \in B^*$:

$$\begin{aligned} f^{**}(\Gamma)(\psi) &= \mathrm{ev}(\psi)(f^{**}(\Gamma)) = \mathrm{ev}(\psi)(\widetilde{\eta_B \circ f}(\Gamma)) = \Gamma((\eta_B \circ f)^\sigma(\mathrm{ev}(\psi))) \\ &= \Gamma(\mathrm{ev}(\psi) \circ \eta_B \circ f). \end{aligned}$$

Now, given $b \in B$:

$$(\mathrm{ev}(\psi) \circ \eta_B)(b) = \mathrm{ev}(\psi)(\eta_B(b)) = \eta_B(b)(\psi) = \psi(b),$$

so

$$f^{**}(\Gamma)(\psi) = \Gamma(\mathrm{ev}(\psi) \circ \eta_B \circ f) = \Gamma(\psi \circ f),$$

as required.

We know that ϵ_A is defined in each case as $\widetilde{\mathrm{id}_A}$. This is a normal unital *-homomorphism because id_A is a unital *-homomorphism. For all $\phi \in A_*$ and $\Gamma \in A^{**}$ we have

$$\phi(\epsilon_A(\Gamma)) = \phi(\widetilde{\mathrm{id}_A}(\Gamma)) = \Gamma(\mathrm{id}_A^\sigma(\phi)) = \Gamma(\phi).$$

□

Convention 2.33. Since the only Banach *-algebras that we are interested in here are C*-algebras, we can now ignore the distinction between $\triangleleft$ and $\triangleright$ in cases where we are not directly applying the definition and only using the existence of the multiplication, in which case we can use juxtaposition or $\cdot$ to represent the multiplication. □

2.5 Gel'fand Duality for W*-algebras

Here we describe Gel'fand duality for commutative W*-algebras. The categorical aspects were developed independently by Dmitri Pavlov [24] and the author.[6] The reader can consult [24] for the proofs as we only give a sketch here.

If (X, Σ) is a measurable space, we can define $\mathcal{L}^\infty(X, \Sigma)$ to be the *-algebra of bounded measurable $\mathbb{C}$-valued functions on X. This is a closed *-subalgebra of $\ell^\infty(X)$ and therefore a commutative unital C*-algebra. On maps, we have $\mathcal{L}^\infty(f)(b) = b \circ f$ and this is a functor from the category of measurable spaces **Mble** to $\mathbf{CC^*Alg}^{\mathrm{op}}$, and $\mathcal{L}^\infty(f)$ is always σ-normal.

If (X, Σ, ν) is a measure space, we can define

$$\mathcal{I}(\nu) = \{a \in \mathcal{L}^\infty(X) \mid \nu(a^{-1}(\mathbb{C} \setminus \{0\})) = 0\}$$

i.e. the functions supported on a set of measure zero. Then $L^\infty(X, \Sigma, \nu) = \mathcal{L}^\infty(X, \Sigma)/\mathcal{I}(\nu)$ is a commutative unital C*-algebra. If f is a nullset-reflecting measurable map, then $L^\infty(f)$, defined as for $\mathcal{L}^\infty(f)$, is well-defined. However, in general it only produces σ-normal *-homomorphisms, not normal ones, a matter we address later.

In order to get $L^\infty(X, \Sigma, \nu)$ to be a W*-algebra, we need (X, Σ, ν) to be a *localizable* measure space [9, 211G, 243G]. However, localizable measure spaces are still too wild to produce a duality – one does not get that L^∞ is a full functor.

The map from $A \in \mathbf{CW^*Alg}^{\mathrm{op}}$ to measure spaces is provided by taking the Gel'fand spectrum $\mathrm{Spec}(A)$, and equipping it with the σ-algebra of sets with the Baire property, which in this situation can equivalently be described as those symmetrically differing from a Borel set by a meagre set, or from a clopen set by a meagre set. It is possible to put a localizable measure on $\mathrm{Spec}(A)$ whose null sets are the meagre sets, so we choose one of these arbitrarily.

What we require on our measure spaces is that they be compact [10, 342A (c)], complete [9, 211A] and strictly localizable[7] [9, 211E]. Define $\mathcal{M}eas$ to have these measure spaces as objects and nullset-reflecting measurable maps as morphisms. This ensures that for any normal unital *-homomorphism $g : L^\infty(Y) \to L^\infty(X)$ we get a nullset-reflecting measurable map $f : X \to Y$ such that $L^\infty(f) = g$ [10, 343B (v)]. Furthermore the measure space $\mathrm{Spec}(A)$ is compact, complete and strictly localizable, and the completeness and strict localizability on the one hand and compactness on the other provide a morphism $(X, \Sigma, \nu) \to \mathrm{Spec}(L^\infty(X, \Sigma, \nu))$ and its inverse.

[6] Unfortunately the author's work was in a chapter of his thesis that he was forced to remove by some members of the thesis committee.

[7] We in fact must alter Fremlin's definition slightly and require that the elements of a decomposition have strictly positive measure. This is to take care of the empty set as a measure space correctly.

As we have defined it, L^∞ is not faithful. We define an equivalence relation on $\mathcal{M}eas(X, Y)$

$$f \sim g \Leftrightarrow \forall T \in \Sigma_Y . \nu_X(f^{-1}(T) \triangle g^{-1}(T)) = 0.$$

We then form **Meas** by quotienting the hom sets by this relation. Then the functor L^∞ is faithful. This relation is strictly coarser than equality almost everywhere, in general [10, 343I] [21].

Although $L^\infty(f)$ is in general only σ-normal and not normal, this problem does not occur if f maps from a compact localizable space to a strictly localizable space, because this would contradict [11, 451Q][8]. So we do not need add anything to nullset-reflecting when working with **Meas**.

Once this is all set up, L^∞ and Spec form an equivalence $\mathbf{Meas} \simeq \mathbf{CW^*Alg}^{\mathrm{op}}$. As well as the "Gel'fand realization" Spec(A) of a commutative W*-algebra A, there is also a "Maharam realization". For each A there is a measure space Y such that $L^\infty(Y) \cong A$ such that $Y = \coprod_{i \in I}(2^{\kappa_i}, \widehat{\mathcal{B}o(2^{\kappa_i})}, \hat{\nu}_\kappa)$ where ν_κ is the measure representing an independent sequence of κ_i fair Bernoulli trials, κ_i being a cardinal [10, 332B]. Since Y is compact, complete and strictly localizable, if we obtain this space for $X \in \mathbf{Meas}$, we have an isomorphism $X \cong Y$ because L^∞ is an equivalence.

3 $C(2^\omega)^{**}$ in Terms of Measure Spaces

It seems common that, when introduced to $C([0,1])^{**}$, people think it must actually be something familiar. Usually people expect that it is one of $L^\infty([0,1])$, $\ell^\infty([0,1])$, or $\mathcal{L}^\infty([0,1])$. It is none of these things, and the last one is not even a W*-algebra. [9]

However, with the techniques developed in this subsection we will eventually be able to find a measure space (X, Σ_X, ν_X) such that $C([0,1])^{**} \cong L^\infty(X)$, *i.e.* in the terminology of the previous section, we will find its "Maharam realization".

We first require some general notions for normal states on W*-algebras.

Definition 3.1. *If ϕ is a normal state on a W*-algebra A, the* null projection *n_ϕ is defined as*

$$n_\phi = \bigvee \{p \in \mathrm{Proj}(A) \mid \phi(p) = 0\},$$

and normality of ϕ implies $\phi(n_\phi) = 0$, which means it is the largest projection mapping to 0. *The* support projection *p_ϕ is*

$$p_\phi = \bigwedge \{p \in \mathrm{Proj}(A) \mid \phi(p) = 1\}.$$

Normality of ϕ implies $\phi(p_\phi) = 1$, and we have $p_\phi = 1 - n_\phi$.

If ϕ, ψ are normal states, we say that they are orthogonal *iff p_ϕ and p_ψ are orthogonal, and write $\phi \perp \psi$.*

An orthogonal family of normal states *is a family $\{\phi_i\}_{i \in I}$ such that $i \neq j \in I$ implies $\phi_i \perp \phi_j$.* □

[8] 451P in earlier editions.

[9] See [19, §8.3 Theorem 14 (ii)] for this mistake being made by an author I hold in high esteem. The error was pointed out in [16].

The following lemma is well-known, and used for example in [33, VII Theorem 2.7] to prove the existence of a faithful semi-finite normal weight on any W*-algebra.

Lemma 3.2. *Let A be a W*-algebra. Orthogonal families of normal states form a poset under the relation of extension. An orthogonal family $\{\phi_i\}_{i\in I}$ of normal states on A is maximal iff $\bigvee_{i\in I} p_{\phi_i} = 1$. Every orthogonal family of normal states extends to a maximal family.*

Proof. It is easy to verify that extension of families is reflexive, antisymmetric and transitive, so this is left to the reader. We prove the characterization of maximal orthogonal families of normal states as follows. Let $\{\phi_i\}_{i\in I}$ be an orthogonal family of normal states such that $\bigvee_{i\in I} p_{\phi_i} = 1$, and suppose that it could be extended by a normal state ψ, orthogonal to ϕ_i for all $i \in I$. But since $p_\psi \perp p_{\phi_i}$ for all $i \in I$, $p_\psi \perp 1$ so $p_\psi = 0$, a contradiction.

In the other direction, suppose that $\{\phi_i\}_{i\in I}$ is a maximal orthogonal family of normal states. Let $p = \bigvee_{i\in I} p_{\phi_i}$, and suppose for a contradiction that $p < 1$, and define $q = 1 - p$. If $\phi(q) = 0$ for all normal states ϕ, then this is true for all $\psi \in A_*$, contradicting the fact that A is isomorphic to the dual of A_*. So there exists a state ϕ such that $\phi(q) > 0$. We define $\psi : A \to \mathbb{C}$ by $\psi(a) = \frac{\phi(qaq)}{\phi(q)}$. It is then easy to verify that ψ is a normal state on A such that $\psi(q) = 1$, and therefore $\psi \perp \phi_i$ for all $i \in I$, contradicting the maximality of $\{\phi_i\}_{i\in I}$.

To prove that each orthogonal family of normal states extends a maximal family, all we need is Zorn's lemma, and the fact that the union of a chain of orthogonal families of normal states is an orthogonal family of normal states, which is easily proved. □

We require the following lemma to simplify some algebra in the next proof, which we have made external because it is used twice.

Lemma 3.3. *Let A, B be involutive vector spaces, and $f, g : A \to B$ $\mathbb{C}$-linear maps. If f and g agree on self-adjoint elements of A, then $f = g$. This also holds if f, g are both $\mathbb{C}$-antilinear, because a $\mathbb{C}$-antilinear map $A \to B$ is a $\mathbb{C}$-linear map $A \to \overline{B}$.*

Proof. Let $a \in A$. We can write $a_\Re = \frac{a+a^*}{2}$ and $a_\Im = \frac{a-a^*}{2i}$, which are both self-adjoint, and then $a = a_\Re + ia_\Im$. Then

$$f(a) = f(a_\Re + ia_\Im) = f(a_\Re) + if(a_\Im) = g(a_\Re) + ig(a_\Im) = g(a_\Re + ia_\Im) = g(a).$$

□

We can now relate, for any compact Hausdorff space X, the algebras $C(X)^{**}$ and $\mathcal{L}^\infty(X)$, where we use the Baire σ-algebra $\mathcal{B}a(X)$.

Definition 3.4. *Let X be a compact Hausdorff space, which we equip with the Baire σ-algebra when it is being treated as a measurable space. Let $\mathcal{M}(X)$ denote the Banach space of bounded complex-valued measures on X with its total variation norm [10, 326Y (e),(m)]. Then define $s_X : \mathcal{M}(X) \to C(X)^*$, by taking for each $\nu \in \mathcal{M}(X)$ and $a \in C(X)$:*

$$s_X(\nu)(a) = \int_X a \, \mathrm{d}\nu.$$

By the complex version of the Riesz representation theorem, this is an isometric isomorphism of Banach spaces. We use $t_X : C(X)^* \to \mathcal{M}(X)$ for the inverse.

We can therefore define a map $\beta_X : \mathcal{L}^\infty(X) \to C(X)^{**}$ by taking for each $a \in \mathcal{L}^\infty(X)$ and $\phi \in C(X)^*$:

$$\beta_X(a)(\phi) = \int_X a \, \mathrm{d}t_X(\phi),$$

so in particular for $\nu \in \mathcal{M}(X)$ we have

$$\beta_X(a)(s_X(\nu)) = \int_X a \, \mathrm{d}\nu.$$

*The map β_X is a σ-normal unital *-homomorphism. The diagram*

$$\begin{tikzcd} C(X) \arrow[r, hook] \arrow[dr, "\eta_X"'] & \mathcal{L}^\infty(X) \arrow[d, "\beta_X"] \\ & C(X)^{**} \end{tikzcd} \tag{3.5}$$

commutes, i.e. *β_X agrees with η_X when restricted to continuous functions.*

Proof. We first have to show that $\beta_X(a)$ is a bounded linear map $C(X)^* \to \mathbb{C}$. Since the maps s_X, t_X arising from the Riesz representation theorem are linear, and integration is linear in the measure, we have for all $a \in \mathcal{L}^\infty(X)$, $\alpha \in \mathbb{C}$ and $\phi_1, \phi_2 \in C(X)^*$

$$\begin{aligned} \beta_X(a)(\alpha\phi_1 + \phi_2) &= \int_X a \, \mathrm{d}t_X(\alpha\phi_1 + \phi_2) \\ &= \int_X a \, \mathrm{d}\alpha t_X(\phi_1) + t_X(\phi_2) \\ &= \alpha \int_X a \, \mathrm{d}t_X(\phi_1) + \int_X a \, \mathrm{d}t_X(\phi_2) \\ &= \alpha\beta_X(a)(\phi_1) + \beta_X(a)(\phi_2), \end{aligned}$$

so $\beta_X(a)$ is linear. If a is a bounded real-valued measurable function and ν a bounded real-valued signed measure, then $|\int_X a \, \mathrm{d}\nu| \leq \|a\|\|\nu\|$. Therefore, by splitting into real and imaginary parts we have $|\int_X a \, \mathrm{d}\nu| \leq 4\|a\|\|\nu\|$ when[10] $a \in \mathcal{L}^\infty(X)$ and $\nu \in \mathcal{M}(X)$. Since t_X is an isometry, this proves that $\beta_X(a)$ is a bounded linear functional on $C(X)^*$.

The linearity of β_X itself follows from the linearity of integration, and the σ-normality of β_X is a standard application of the dominated convergence theorem.

By the definition of s_X and t_X coming from the Riesz representation theorem, for all $a \in C(X)$ we have

$$\beta_X(a)(\phi) = \int_X a \, \mathrm{d}t_X(\phi) = \phi(a) = \eta_X(a)(\phi),$$

which proves that (3.5) commutes, and also that β_X is unital.

[10]The constant 4 can be improved to 1, but we do not require this so it isn't worth the effort.

We aim to show that β_X preserves the $*$, which is the same as $\beta_X \circ \text{-}^* = \text{-}^* \circ b_X$. As these are both $\mathbb{C}$-antilinear maps $\mathcal{L}^\infty(X) \to C(X)^{**}$, by Lemma 3.3 it suffices to very this on self-adjoint (*i.e.* real-valued) elements $a \in \mathcal{L}^\infty(X)$. So we aim to show that $\beta_X(a^*) = \beta_X(a)^*$. Since these are $\mathbb{C}$-linear maps $C(X)^* \to \mathbb{C}$, by Lemma 3.3 it suffices to verify this for self-adjoint $\phi \in C(X)^*$. We then have

$$\beta_X(a^*)(\phi) = \beta_X(a)(\phi) = \int_X a \, \mathrm{d}t_X(\phi),$$

and

$$\beta_X(a)^*(\phi) = \overline{\beta_X(a)(\phi^*)} = \overline{\beta(a)(\phi)} = \overline{\int_X a \, \mathrm{d}t_X(\phi)},$$

and these two are equal, because the integral of a real-valued function by a real-valued signed measure is real. This proves that β_X preserves the $*$.

We show that β_X preserves multiplication as follows. We aim to show that for all $a, b \in \mathcal{L}^\infty(X)$, $\beta_X(ab) = \beta_X(a) \triangleleft \beta_X(b)$. We start by evaluating the right hand side at an arbitrary $\phi \in C(X)^*$:

$$\begin{aligned}(\beta_X(a) \triangleleft \beta_X(b))(\phi) &= \beta_X(a)(\beta_X(b) \triangleleft \phi) \\ &= \int_X a \, \mathrm{d}t_X(\beta_X(b) \triangleleft \phi)\end{aligned}$$

We also evaluate the left hand side, but first recall that for $\nu \in \mathcal{M}(X)$, if $b \in \mathcal{L}^1(X, \nu)$ we can define the bounded complex measure $b \cdot \nu$

$$b \cdot \nu(S) = \int_X \chi_S \, \mathrm{d}\nu,$$

where $S \in \mathcal{B}a(X)$. It follows by a routine linearity and dominated convergence argument that for all $a \in \mathcal{L}^\infty(X)$ we have

$$\int_X a \, \mathrm{d}b \cdot \nu = \int_X ab \, \mathrm{d}\nu.$$

So

$$\beta_X(ab)(\phi) = \int_X ab \, \mathrm{d}t_X(\phi) = \int_X a \, \mathrm{d}b \cdot t_X(\phi).$$

In order to prove that this is equal to $\int_X a \, \mathrm{d}t_X(\beta_X(b) \triangleleft \phi)$, for all $a \in \mathcal{L}^\infty(X)$, it suffices to verify it for all $a \in C(X)$. So let $a \in C(X)$, and by the Riesz representation theorem

$$\int_X a \, \mathrm{d}t_X(\beta_X(b) \triangleleft \phi) = (\beta_X(b) \triangleleft \phi)(a) = \beta_X(b)(\phi \triangleleft a) = \int_X b \, \mathrm{d}t_X(\phi \triangleleft a),$$

While

$$\int_X a \, \mathrm{d}b \cdot t_X(\phi) = \int_X ab \, \mathrm{d}t_X(\phi) = \int_X ba \, \mathrm{d}t_X(\phi) = \int_X b \, \mathrm{d}a \cdot t_X(\phi).$$

Again, to verify this for all $b \in \mathcal{L}^\infty(X)$, it suffices to verify it for all $b \in C(X)$, so let $b \in C(X)$, and by the Riesz representation theorem

$$\int_X b \, \mathrm{d}t_X(\phi \triangleleft a) = (\phi \triangleleft a)(b) = \phi(ab) = \int_X ab \, \mathrm{d}t_X(\phi),$$

as required. All together, this shows that β_X is a σ-normal unital *-homomorphism.

□

Proposition 3.6. *Let X be a compact Hausdorff space and ν a Baire probability measure. Recall that we define*

$$\mathcal{I}(\nu) = \{a \in \mathcal{L}^\infty(X) \mid \nu(a^{-1}(\mathbb{C} \setminus \{0\})) = 0\}$$

*and this is a closed *-ideal and $L^\infty(X, \nu) = \mathcal{L}^\infty(X)/\mathcal{I}(\nu)$ which is a W*-algebra with predual $L^1(X, \nu)$. Writing $q : \mathcal{L}^\infty(X) \to L^\infty(X)$, and q_0 for its restriction to $C(X)$ we obtain a commuting diagram*

$$\begin{tikzcd} C(X) \arrow[r, hook] \arrow[drr, "q_0"'] & \mathcal{L}^\infty(X) \arrow[r, "\beta_X"] \arrow[dr, "q"] & C(X)^{**} \arrow[d, "\tilde{q}_0"] \\ & & L^\infty(X, \nu) \end{tikzcd} \tag{3.7}$$

Let $\phi = \mathrm{ev}(s_X(\nu))$ be the normal state on $C(X)^{**}$ defined by ν (where ev *is the evaluation map $C(X)^* \to C(X)^{***}$). The kernel projection of $\tilde{q}_0$ (see Proposition 2.17) is exactly the null projection n_ϕ of ϕ (recall Definition 3.1). Therefore $\tilde{q}_0$ restricts to an isomorphism $p_\phi C(X)^{**} p_\phi \cong L^\infty(X, \nu)$.*

Proof. The fact that $L^\infty(X)$ is a W*-algebra with predual $L^1(X)$ is a standard fact [9, 243G Theorem, 243K] [32, III, Theorem 1.2 (iii)].

By definition the left triangle of (3.7) commutes, and the outer triangle commutes because $\beta_X|_{C(X)} = \eta_X$ (by (3.5)) and the universal property diagram for η_X (2.31). Therefore the right triangle commutes when q and β_X are restricted to $C(X)$. The map $\tilde{q}_0 \circ \beta_X$ is σ-normal, because $\tilde{q}_0$ is normal (Definition 2.30) and β_X is σ-normal (Definition 3.4). The map q is σ-normal because $\mathcal{I}(\nu)$ is closed under bounded monotone suprema of sequences. Since every bounded Baire function on a compact Hausdorff space can be obtained by taking iterated monotone increasing or decreasing limits of sequences starting with continuous functions, this proves that $q = \tilde{q}_0 \circ \beta_X$ on all of $\mathcal{L}^\infty(X)$.

Taking $\phi = \mathrm{ev}(s_X(\nu))$, the normal state on $C(X)^{**}$ defined by ν, we aim to show that the kernel projection of $\tilde{q}_0$ is the null projection of ϕ. It suffices to show that for any projection $P \in \mathrm{Proj}(C(X)^{**})$ we have $\phi(P) = 0$ iff $\tilde{q}_0(P) = 0$, because then the null projection and the kernel projection are the least upper bound of the same set of projections.

Define $\psi : L^\infty(X, \nu) \to \mathbb{C}$ to be the normal state on $L^\infty(X, \nu)$ defined by ν, so $\psi([a]) = \int_X a \, \mathrm{d}\nu$ for $a \in \mathcal{L}^\infty(X)$. It is obvious that $q_0^\sigma(\psi) = \psi \circ q_0 = s_X(\nu)$. Now, if $\tilde{q}_0(P) = 0$, then it follows that $\psi(\tilde{q}_0(P)) = 0$. By definition

$$\phi(P) = \mathrm{ev}(s_X(\nu))(P) = P(s_X(\nu)) = P(q_0^\sigma(\psi)) = \psi(\tilde{q}_0(P)) = 0,$$

using (2.32) for q_0.

In the other direction, suppose that $\phi(P) = 0$, and we aim to show that $\tilde{q}_0(P) = 0$. To prove this, since $L^1(X, \nu)$ is a predual for $L^\infty(X, \nu)$, it suffices to show that for all $a \in \mathcal{L}^1(X, \nu)$ that $\int_X \tilde{q}_0(P) a \, \mathrm{d}\nu = 0$. In turn, by applying the dominated convergence theorem and the fact that every ν-integrable Baire function occurs by iterating pointwise limits of continuous functions on X, it suffices to show that for all $a \in C(X)$, $\int_X \tilde{q}_0(P) a \, \mathrm{d}\nu = 0$. If we let $\psi_a : L^\infty(X, \nu) \to \mathbb{C}$ be the normal linear functional defined by $\psi_a(b) = \int_X ab \, \mathrm{d}\nu$, then we have by (2.32)

$$\int_X \tilde{q}_0(P) a \, \mathrm{d}\nu = \psi_a(\tilde{q}_0(P)) = P(q_0^\sigma(\psi_a))$$

and we see that for all $b \in C(X)$

$$(s_X(\nu) \triangleleft a)(b) = s_X(\nu)(ab) = \int_X ab \, \mathrm{d}\nu = \psi_a(q_0(b)) = q_0^\sigma(\psi_a)(b),$$

so continuing the previous equation and using the Arens product identities (Definition 2.22)

$$\begin{aligned}\int_X \tilde{q_0}(P) a \, \mathrm{d}\nu &= P(q_0^\sigma(\psi_a)) = P(s_X(\nu) \triangleleft a) = P(s_X(\nu) \triangleright \mathrm{ev}(a)) \\ &= (\mathrm{ev}(a) \triangleright P)(s_X(\nu)) = \phi(\mathrm{ev}(a) \triangleright P).\end{aligned}$$

By the Cauchy-Schwarz inequality for states [5, 2.1.2 (2)] (this time writing the product on $C(X)^{**}$ by juxtaposition):

$$|\phi(\mathrm{ev}(a)P)|^2 \leq \phi(\mathrm{ev}(a)\mathrm{ev}(a)^*)\phi(P^2) = \phi(\mathrm{ev}(a)\mathrm{ev}(a)^*)\phi(P).$$

Since by the initial assumption $\phi(P) = 0$, we have $\phi(P\mathrm{ev}(a)) = 0$, so $\int_X \tilde{q_0}(P) a \, \mathrm{d}\nu = 0$. As discussed, since this holds for all continuous a this implies $\tilde{q_0}(P) = 0$.

It then follows that $\tilde{q_0}$ restricts to an isomorphism $p_\phi C(X)^{**} p_\phi \cong L^\infty(X, \nu)$ by Proposition 2.17. □

Proposition 3.8. *Let X be a compact Hausdorff space and ν_1, ν_2 be Baire probability measures on X, and write ϕ_1, ϕ_2 for the corresponding normal linear functionals on $C(X)^{**}$ defined by the Riesz representation theorem, so $\phi_i = \mathrm{ev}(s_X(\nu_i))$ (for $i \in \{1, 2\}$, where $\mathrm{ev} : C(X)^* \to C(X)^{***}$ is the evaluation embedding). We have $\phi_1 \perp \phi_2$ iff there exists $S \in \mathcal{B}a(X)$ such that $\nu_1(S) = 1$ and $\nu_2(S) = 0$.*

Proof. Suppose there is such an $S \in \mathcal{B}a(X)$ such that $\nu_1(S) = 1$ and $\nu_2(S) = 0$. Then $\beta_X(\chi_S)$ is a projection in $C(X)^{**}$, and for all $i \in \{1, 2\}$:

$$\phi_i(\beta_X(\chi_S)) = \mathrm{ev}(s_X(\nu_i))(\beta_X(\chi_S)) = \beta_X(\chi_S)(s_X(\nu_i)) = \int_X \chi_S \, \mathrm{d}\nu_i = \nu_i(S)$$

So $\phi_1(\beta_X(\chi_S)) = 1$ and $\phi_2(\beta_X(\chi_S)) = 0$. So (by Definition 3.1) $p_{\phi_1} \leq \beta_X(\chi_S)$ and $p_{\phi_2} \leq 1 - \beta_X(\chi_S)$, and therefore $\phi_1 \perp \phi_2$.

Now suppose that $\phi_1 \perp \phi_2$. Define $\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$. The map $q : \mathcal{L}^\infty(X) \to L^\infty(X, \nu)$ is surjective, as it is a quotient mapping. For a set $S \in \mathcal{B}a(X)$, we have $\nu(S) = 0$ implies $\nu_1(S) = \nu_2(S) = 0$, by positivity, so $\nu_1, \nu_2 \ll \nu$. For each probability measure ν' such that $\nu' \ll \nu$ and we can define a normal state $s_X^\nu(\nu')$ on $L^\infty(X, \nu)$ by

$$s_X^\nu(\nu')(q(a)) = \int_X a \, \mathrm{d}\nu',$$

where the well-definedness of this follows from $\nu' \ll \nu$.

We define $P_1 = \mathrm{supp}(\phi_1)$ and by surjectivity of the quotient map there exists $S \in \mathcal{B}a(X)$ such that $q(\chi_S) = \tilde{q_0}(P_1)$. For each $i \in \{1, 2\}$ we have

$$\nu_i(S) = \int_X \chi_S \, \mathrm{d}\nu_i = s_X^\nu(\nu_i)(q(\chi_S)) = s_X^\nu(\nu_i)(\tilde{q_0}(P_1)) = P_1(q_0^\sigma(s_X^\nu(\nu_i))).$$

Then for each $a \in C(X)$ we have

$$q_0^\sigma(s_X^\nu(\nu_i))(a) = s_X^\nu(\nu_i)(q_0(a)) = \int_X a \,\mathrm{d}\nu_i = s_X(\nu_i)(a),$$

so

$$\nu_i(S) = P_1(s_X(\nu_i)) = \phi_i(P_1),$$

and since $\phi_1 \perp \phi_2$ we have $\nu_1(S) = \phi_1(P_1) = 1$ and $\nu_2(S) = \phi_2(P_1) = 0$. □

Theorem 3.9. *Let 2^ω be the Cantor space, ν_c the probability measure defined by a sequence of independent fair coin flips. Then*

$$C(2^\omega)^{**} \cong \ell^\infty(2^\omega) \times \prod_{i \in 2^\omega} L^\infty(2^\omega, \nu_c)$$

If we let ν_d be the discrete counting measure on $\mathcal{P}(2^\omega)$, we therefore have an isomorphism between $C(2^\omega)$ and $L^\infty(X)$, where the measure space X is defined as

$$X = (2^\omega, \mathcal{P}(2^\omega), \nu_d) + (2^\omega \times 2^\omega, \widehat{\mathcal{B}o(2^\omega)} \otimes \mathcal{P}(2^\omega), \hat{\nu_c} \otimes \nu_d),$$

where $\widehat{\mathcal{B}o(2^\omega)}$ is the completion of the Borel sets of 2^ω for the measure ν_c and $\otimes$ is the c.l.d. product of Fremlin [9, Definition 251F].

For any uncountable compact metric space Y, $C(Y)^{**} \cong C(2^\omega)^{**}$, so this also describes $C(Y)^{**}$.

Proof. We first find an orthogonal family of normal states on $C(2^\omega)^{**}$. We start with $(\delta_x)_{x\in 2^\omega}$, the δ-measures. If $x_1, x_2 \in 2^\omega$ are distinct points, then $\delta_{x_1}(\{x_1\}) = 1$ and $\delta_{x_2}(\{x_1\}) = 0$, so the corresponding family of normal functionals on $C(X)^{**}$ is orthogonal (Proposition 3.8).

By Lemma 3.2, this family extends to a maximal one, so there is a family of Baire probability measures $(\nu_i)_{i\in I}$ on 2^ω that are each orthogonal to every state defined by $(\delta_x)_{x\in 2^\omega}$ and mutually orthogonal. It follows that for all $x \in 2^\omega$, there exists a Baire set $S \in \mathcal{B}a(2^\omega)$ such that $\delta_x(S) = 1$ and $\nu_i(S) = 0$. This implies $x \in S$ so $\nu_i(\{x\}) = 0$, *i.e.* each of the measures vanishes on singletons. It follows [18, Theorem 17.41] that for each $i \in I$, there is a measure-preserving Borel isomorphism $(2^\omega, \nu_i) \cong (2^\omega, \nu_c)$, where ν_c is the fair coin-flip measure restricted to Borel sets (which are the same as Baire sets for 2^ω). So it remains to prove that we can take $I = 2^\omega$.

We have an upper bound on the cardinality of I. The state space of $C(2^\omega)$ is metrizable and separable, so has cardinality $\leq |\mathbb{N}^\mathbb{N}| = 2^{\aleph_0}$. Therefore there cannot be $> 2^{\aleph_0}$ orthogonal normal states on $C(2^\omega)^{**}$ because there are only $2^{\aleph_0}$ normal states at all.

We now find a family of Baire measures on 2^ω of size continuum. For each $\alpha \in (0, 1)$, let ν_α be the Baire measure on 2^ω representing a sequence of independent Bernoulli trials taking the value 0 with probability α and 1 with probability $1-\alpha$ (so $\nu_c = \nu_{\frac{1}{2}}$). Each of these measures assigns probability 0 to every singleton, so they are all orthogonal to all the measures in $(\delta_x)_{x\in 2^\omega}$. Consider the sets

$$S_\alpha = \left\{ a \in 2^\omega \;\middle|\; \lim_{n\to\infty} \sum_{i=0}^{n} \frac{a(n)}{n+1} = \alpha \right\}$$

By the strong law of large numbers [8, VII.8 Theorem 1], these are Borel subsets of 2^ω and $\nu_\alpha(S_\alpha) = 1$ for each $\alpha \in (0,1)$, but the two occurrences of α have to match. Since the limit of a sequence in $[0,1]$ has at most one value, it is clear that if $\alpha \neq \alpha'$ then $S_\alpha \cap S_{\alpha'} = \emptyset$, so $\nu_{\alpha'}(S_\alpha) = 0$. So $(\nu_\alpha)_{\alpha\in(0,1)}$ is a family of orthogonal measures of cardinality $2^{\aleph_0}$ (Lemma 3.2). By extending it to a maximal family, we get a maximal orthogonal family of Baire measures $(\delta_x)_{x\in 2^\omega} \times (\nu_i)_{i\in 2^\omega}$.

We write $(p^{\delta_x})_{x\in 2^\omega}$ for the support projections of the normal states corresponding to $(\delta_x)_{x\in 2^\omega}$ and $(p^{\nu_i})_{i\in 2^\omega}$ for the support projections of the normal states corresponding to $(\nu_i)_{i\in 2^\omega}$. By Theorem 2.12 we have isomorphisms

$$\begin{aligned}
C(2^\omega)^{**} &\cong \prod_{x\in 2^\omega} p^{\delta_x} C(2^\omega)^{**} p^{\delta_x} \times \prod_{i\in 2^\omega} p^{\nu_i} C(2^\omega)^{**} p^{\nu_i} && \text{Theorem 2.12}\\
&\cong \prod_{x\in 2^\omega} L^\infty(2^\omega,\delta_x) \times \prod_{i\in 2^\omega} L^\infty(2^\omega,\nu_i) && \text{Proposition 3.6}\\
&\cong \prod_{x\in 2^\omega} \mathbb{C} \times \prod_{i\in 2^\omega} L^\infty(2^\omega,\nu_c) && \text{[18, Theorem 17.41]}\\
&\cong \ell^\infty(2^\omega) \times \prod_{i\in 2^\omega} L^\infty(2^\omega,\nu_c).
\end{aligned}$$

To get this in the form of L^∞ of a complete compact strictly localizable measure space, it suffices to take $\coprod_{x\in 2^\omega}(\{x\},\delta_x) + \coprod_{i\in 2^\omega}(2^\omega,\hat{\nu_c})$, where $\hat{\nu_c}$ is the (Lebesgue) completion of the Borel measure ν_c. This is isomorphic to $(2^\omega,\nu_d) + (2^\omega \times 2^\omega, \nu_d \otimes \hat{\nu_c})$, where we use $\otimes$ to represent Fremlin's c.l.d. product [9, Definition 251F], which by definition is complete, and is strictly localizable and compact if the spaces being multiplied are [9, 251N Proposition] [10, Proposition 342G (e)]. It follows from [9, 251N Proposition] that the c.l.d. product of a space by the counting measure on X is the same as the X-fold coproduct [9, 251X (h)].

Finally, if X is an uncountable compact metrizable space, it is an uncountable Polish space, so there is a Borel isomorphism $f : X \to 2^\omega$ [18, Theorem 15.6]. For each $\nu \in \mathcal{M}(X)$, we can define the pushforward $f_*(\nu) \in \mathcal{M}(2^\omega)$ by $f_*(\nu)(S) = \nu(f^{-1}(S))$, and this gives an isometric isomorphism of Banach spaces $f_* : \mathcal{M}(X) \to \mathcal{M}(2^\omega)$. Using the Riesz representation theorem and dualizing gives an isometric isomorphism $f_*^\sigma : C(2^\omega)^{**} \to C(X)^{**}$. If two commutative C*-algebras are isometrically isomorphic, then they are *-isomorphic, by the Banach-Stone theorem [3, IV Theorem 2.1].[11] $\square$

In the above, $C(2^\omega)^{**}$ was expressed as an (uncountable) union of standard Lebesgue spaces. We remark that not all W*-algebras can be expressed as such. Consider $(\ell^\infty)^{**}$. The predual of it is $(\ell^\infty)^*$, and we have $\mathrm{Spec}(\ell^\infty) \cong \beta(\mathbb{N})$. In a stonean space, such as $\beta(\mathbb{N})$, every convergent sequence is eventually constant, so if a point has a countable neighbourhood base, it must be an isolated point. Therefore $\beta(\mathbb{N}) \setminus \mathbb{N}$ is nowhere first-countable, so there is an embedding $2^{\aleph_1} \hookrightarrow \beta(\mathbb{N}) \setminus \mathbb{N} \subseteq \beta(\mathbb{N})$, by the Čech-Pospíšil theorem [15, 7.19 Theorem], and therefore an embedding $2^{\aleph_1} \hookrightarrow \mathrm{Spec}(\ell^\infty)$. We can put the independent coin-flipping measure on the copy of $2^{\aleph_1}$ and have the complement of it be of measure

[11] In fact this dual map is a *-isomorphism, but since we only need *an* isomorphism, we take this short cut.

zero. This gives a probability measure on $\mathrm{Spec}(\ell^\infty(X))$ of Maharam type $\aleph_1$, so cannot be expressed using standard Lebesgue spaces, which have countable Maharam type.

In fact it is possible to improve the above and show that if A is a commutative W*-algebra then there is a state $\phi \in A^*$ such that, letting p_ϕ be the support projection, the complete Boolean algebra $\mathrm{Proj}(p_\phi A^{**} p_\phi)$ has Maharam type $\geq 2^{\aleph_0}$. However, we leave this out for reasons of space.

4 Monadicity of W*-algebras

The construction of the comonad in Section 5 uses the fact that commutative W*-algebras are monadic over unital commutative C*-algebras. Since it is useful to have around, we will prove this for the noncommutative case as well.

For the convenience of the reader, we state here some different types of coequalizer that we will be using.

Definition 4.1. *In a category $\mathcal{C}$, a* parallel pair *is a pair of maps $f, g : X \to Y$,* i.e. *with the same domain and codomain. A* reflexive pair *is a parallel pair $f, g : X \to Y$ such that there exists a common section/right inverse $r : Y \to X$,* i.e. *$f \circ r = \mathrm{id}_Y = g \circ r$. A* reflexive coequalizer *is a coequalizer $q : Y \to Q$ of a reflexive pair.* □

Definition 4.2. *In a category $\mathcal{C}$, a* split fork *is a diagram*

$$\begin{tikzcd} X \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & Y \arrow[l, "r"'] \arrow[r, shift left, "h"] & Z, \arrow[l, shift left, "s"] \end{tikzcd}$$

such that $h \circ f = h \circ g$, $f \circ r = \mathrm{id}_Y$, $h \circ s = \mathrm{id}_Z$, and $g \circ r = s \circ h$ [20, VI.6]. The first three conditions can be described as (Z, h) is a cocone on (f, g), r is a section of f and s is a section of h. In a split fork, (Z, h) is actually a coequalizer of (f, g), and we call it a split coequalizer *when viewed as such.* □

We first show that $\mathbf{W^*Alg}$ and $\mathbf{CW^*Alg}$ have coequalizers and that reflexive coequalizers are preserved by the forgetful functors to the corresponding categories of positive unital maps.

Proposition 4.3. *Let $f, g : A \to B$ be a parallel pair in $\mathbf{W^*Alg}$. Define*

$$N = \{f(a) - g(a) \mid a \in A\} \subseteq B.$$

This is a linear subspace, invariant under $*$. Define I to be the smallest weak- closed *-ideal containing N (possibly the improper ideal B itself). The quotient mapping $q : B \to B/I$ is the coequalizer of f and g in $\mathbf{W^*Alg}$. If B is commutative, so is B/I, so this construction also gives coequalizers in $\mathbf{CW^*Alg}$.*

*If f, g form a reflexive pair with common section $r : B \to A$, then N is already a *-ideal, so I is the weak-* closure of N. In this case, A/I is a coequalizer in $\mathbf{W^*Alg}_{\mathrm{PU}}$, and in $\mathbf{CW^*Alg}_{\mathrm{PU}}$ if A, B are commutative.*

Proof. We start by proving that N is a linear subspace of B. If we have two elements of N, they are of the form $f(a_1) - g(a_1), f(a_2) - g(a_2)$ for $a_1, a_2 \in A$, and so if we take a scalar $\alpha \in \mathbb{C}$ we have:

$$\begin{aligned}\alpha(f(a_1) - g(a_1)) + f(a_2) - g(a_2) &= f(\alpha a_1) - g(\alpha a_1) + f(a_2) - g(a_2) \\ &= f(\alpha a_1 + a_2) - g(\alpha a_1 + a_2),\end{aligned}$$

so N is a linear subspace of B. For any $f(a) - g(a)$

$$f(a) - g(a))^* = f(a)^* - g(a)^* = f(a^*) - g(a^*),$$

so N is invariant under $*$.

Skipping ahead, if $r : B \to A$ is a common section of f, g making them into a reflexive pair, we show that N is a *-ideal. If $b \in B$ and $f(a) - g(a) \in N$, then

$$b(f(a) - g(a)) = bf(a) - bg(a) = f(r(b))f(a) - g(r(b))g(a) = f(r(b)a) - g(r(b)a),$$

so N is closed under left multiplication in B. It follows from the invariance under $*$ that we already proved that N is also closed under right multiplication in B and is a *-ideal. Since multiplication is separately weak-* continuous and the -* operation is weak-* continuous, the weak-* closure $\mathrm{cl}\,(N)$ is also a *-ideal, and is therefore the smallest weak-* closed *-ideal containing N.

Returning to the general case, define

$$J = \left\{ \sum_{i=1}^{n} b_i x b_i' \;\middle|\; a \in N, b_i, b_i' \in B \right\}.$$

Then since B is unital we have $N \subseteq J$, and J is clearly a linear subspace of B, it is invariant under $*$ because N is, and it is closed under left multiplication by elements of B on the left, so is a *-ideal containing N. Since *-ideals are right ideals as well, any *-ideal containing N contains J, so it is the smallest *-ideal containing J, and the weak-* closure $I = \mathrm{cl}\,(J)$ is the smallest weak-* closed *-ideal containing N.

The quotient B/I is a W*-algebra and $q : B \to B/I$ is a normal unital *-homomorphism (Proposition 2.16), and it follows that if B is commutative then so is B/I.

Returning to the reflexive case, suppose C is a W*-algebra and $h : B \to C$ a normal positive unital map such that $h \circ f = h \circ g$. For each $f(a) - g(a) \in N$, we have

$$h(f(a) - g(a)) = h(f(a)) - h(g(a)) = 0,$$

so h vanishes on N, and since it is normal, it vanishes on $I = \mathrm{cl}\,(N)$. It follows that the map $\tilde{h} : B/\mathrm{cl}\,(N) \to C$ defined by $\tilde{h}(q(a)) = h(a)$ is well-defined and linear, and if h is unital, so is $\tilde{h}$ because the unit of B/I is $q(1)$. It is also immediate that $\tilde{h} \circ q = h$. A positive element of B/I is of the form $q(b)^* q(b) = q(b^* b)$ for some $b \in B$ and is therefore the image of a positive element of B. So $\tilde{h}(q(b^* b)) = h(b^* b)$ which is positive because h is a positive map. So $\tilde{h}$ is a PU map.

To prove $\tilde{h}$ is normal, let $(x_i)_{i \in I}$ be a net in B/I converging to $x \in B/I$ in the weak-* topology. Recall that there is a unique central projection $p \in B$ such that $I = pAp$, and that $q|_{p^\perp A p^\perp} : p^\perp A p^\perp \to B/I$ is a normal *-isomorphism of W*-algebras (see Proposition 2.16). Using this isomorphism backwards gives a net $(b_i)_{i\in I}$ in B converging to $b \in B$ in the weak-* topology such that $q(b_i) = x_i$ and $q(b) = x$. Then, since h is normal:

$$\tilde{h}(x_i) = \tilde{h}(q(b_i)) = h(b_i) \to h(b) = \tilde{h}(q(b)) = \tilde{h}(x),$$

so $\tilde{h}$ is weak-* continuous, *i.e.* normal. For the uniqueness part of the universal property, suppose that $k : B/I \to C$ is a normal PU-map such that $k \circ q = h$. Since q is surjective, $k = \tilde{h}$.

Returning again to the general, non-reflexive case, let C be a W*-algebra and $h : B \to C$ a normal unital *-homomorphism such that $h \circ f = h \circ g$. As before, h vanishes on N, and we see that for each $b, b' \in B$ and $x \in N$, we have

$$h(bxb') = h(b)h(x)h(b') = h(b)0h(b') = 0,$$

so h vanishes on J, and by normality, on $I = \mathrm{cl}\,(J)$. We can therefore define $\tilde{h}$ as before on B/I, and it is well-defined, linear, unital, normal and $\tilde{h} \circ q = h$ with essentially the same proofs. Given $b, b' \in B$ we have

$$\tilde{h}(q(b)q(b')) = \tilde{h}(q(bb')) = h(bb') = h(b)h(b') = \tilde{h}(q(b))\tilde{h}(q(b')),$$

so $\tilde{h}$ preserves multiplication, and the proof that it preserves the $*$ is similar. The uniqueness part of the universal property follows from the surjectivity of q again. $\square$

The importance of the latter part of the proposition above comes from the fact that neither $\mathbf{W^*Alg}_{\mathrm{PU}}$ nor $\mathbf{CW^*Alg}_{\mathrm{PU}}$ has coequalizers of all parallel pairs.

Proposition 4.4. *Let A, B be W*-algebras and $f, g : A \to B$ be a reflexive pair (Definition 4.1) with common section $r : B \to A$, such that either (f, g) has a split coequalizer in* $\mathbf{C^*Alg}$ *or* $(\mathrm{Ball}(f), \mathrm{Ball}(g))$ *has a split coequalizer in* $\mathbf{Set}$*. Then N, as defined in Proposition 4.3 is already weak-* closed, so is a W*-ideal,* i.e. *a weak-* closed *-ideal in B.*

The quotient mapping $q : B \to B/N$ is a coequalizer of (f, g) in both $\mathbf{W^*Alg}$ *and* $\mathbf{C^*Alg}$*, and, if the algebras are commutative, in* $\mathbf{CW^*Alg}$ *and* $\mathbf{CC^*Alg}$ *as well. Additionally,* $\mathrm{Ball}(q)$ *is a coequalizer of* $(\mathrm{Ball}(f), \mathrm{Ball}(g))$ *in* $\mathbf{Set}$.

Proof. We already know from Proposition 4.3 that N, as defined there, is a *-ideal, and we get the coequalizer in $\mathbf{W^*Alg}$ (and in $\mathbf{CW^*Alg}$ if B is commutative) by taking $B/\mathrm{cl}\,(N)$, where the closure is in the weak-* topology. So we move straight on to showing that N is already weak-* closed.

If $f, g : A \to B$ in $\mathbf{W^*Alg}$ has a split coequalizer in $\mathbf{C^*Alg}$, we can apply Ball to it to get a split coequalizer in $\mathbf{Set}$ of $(\mathrm{Ball}(f), \mathrm{Ball}(g))$, so we work in that situation

$$\begin{tikzcd} \mathrm{Ball}(A) \arrow[r, shift left=2, "\mathrm{Ball}(f)"] \arrow[r, shift right=2, "\mathrm{Ball}(g)"'] & \mathrm{Ball}(B) \arrow[l, "r" description] \arrow[r, shift left, "h"] & Z, \arrow[l, shift left, "s"] \end{tikzcd}$$

which is to say, $h \circ \mathrm{Ball}(f) = h \circ \mathrm{Ball}(g)$, $\mathrm{Ball}(f) \circ r = \mathrm{id}_{\mathrm{Ball}(B)}$, $h \circ s = \mathrm{id}_Z$, and $\mathrm{Ball}(g) \circ r = s \circ h$.

To avoid tedious repetition, whenever we refer to topological notions on A or B in the rest of the proof, we always mean the weak-* topology. We make the following definitions:

$$\begin{aligned}
&R = \{(f(x), g(x)) \mid x \in \mathrm{Ball}(A)\} \subseteq \mathrm{Ball}(B) \times \mathrm{Ball}(B) \\
&S = R \circ R^{\mathrm{op}} \\
&d : B \times B \to B \\
&d(b_1, b_2) = b_1 - b_2 \\
&D = d(S) = \{b_1 - b_2 \mid \exists a_1, a_2 \in \mathrm{Ball}(A). f(a_1) = b_1, f(a_2) = b_2, g(a_1) = g(a_2)\},
\end{aligned}$$

where we used composition of relations to define S in terms of R.

It follows from a basic fact [1, p. 101, Exercise (SPO) (e)] about split coequalizers in **Set** that S is the equivalence relation defined by h, *i.e.* $h(y_1) = h(y_2)$ iff $(y_1, y_2) \in S$. We start by proving some facts about S. First, observe that since $R = \langle f, g\rangle(B(E))$, R is the continuous image of a compact space and therefore compact. Since the relational converse of a compact relation is compact, and the composite of compact relations is compact, it follows that S is compact. Since R is also the image of a convex set ($\mathrm{Ball}(A)$) under a linear map, it is convex, and therefore S is a convex set, being the relational composite of convex relations.

The rest of the proof relies on the following fact, based on a lemma due to Świrszcz [30, Lemma 4.2.2 (IV)]. If $b_1, b_2 \in S$ and $\alpha \in [0, 1)$, then

$$(b_1, \alpha b_1 + (1 - \alpha) b_2) \in S \text{ implies } (b_1, b_2) \in S. \tag{4.5}$$

We prove it as follows. Let $\alpha_0 = \inf\{\alpha \in [0, 1) \mid (b_1, \alpha b_1 + (1-\alpha) b_2) \in S\}$, which exists by the assumption that any such α exists at all. Define $b = \alpha_0 b_1 + (1 - \alpha_0) b_2$, and since α_0 is an infimum and S is closed, $(b_1, b) \in S$. It follows that there exist $a_1, a \in \mathrm{Ball}(A)$ such that $f(a_1) = b_1$, $f(a) = b$ and $g(a_1) = g(a)$. We can also define $a_2 = r(b_2)$ so that $f(a_2) = b_2$. Then

$$\begin{aligned}
f(\alpha_0 a_1 + (1 - \alpha_0) a_2) &= \alpha_0 b_1 + (1 - \alpha_0) b_2 = b, \\
f(\alpha_0 a + (1 - \alpha_0) a_2) &= \alpha_0 b + (1 - \alpha_0) b_2 = \alpha_0^2 b_1 + (1 - \alpha_0^2) b_2, \\
g(\alpha_0 a_1 + (1 - \alpha_0) a_2) &= \alpha_0 g(a_1) + (1 - \alpha_0) g(a_2) = \alpha_0 g(a) + (1 - \alpha_0) g(a_2) \\
&= g(\alpha_0 a + (1 - \alpha_0) a_2).
\end{aligned}$$

So we have $(b, \alpha_0^2 b_1 + (1 - \alpha_0^2) b_2) \in R$, and therefore it is also in S, and since S is a transitive relation (being the equivalence relation defined by h), we have $(b_1, \alpha_0^2 b_1 + (1 - \alpha_0^2) b_2) \in R$. If $\alpha_0 > 0$ then we would have a contradiction at this point because $\alpha_0^2 < \alpha_0$, so $\alpha_0 = 0$ and therefore $(b_1, b_2) \in S$.

We now show that the equivalence relation defined by N, when restricted to $\mathrm{Ball}(B)$, is the same as S. That is to say, for all $b_1, b_2 \in \mathrm{Ball}(B)$, $(b_1, b_2) \in S$ iff $b_1 - b_2 \in N$. For the forward implication, if $(b_1, b_2) \in S$, then by definition there exist $a_1, a_2 \in \mathrm{Ball}(A)$ such that $f(a_1) = b_1$, $f(a_2) = b_2$ and $g(a_1) = g(a_2)$. So

$$f(a_1 - a_2) - g(a_1 - a_2) = f(a_1) - f(a_2) - (g(a_1) - g(a_2)) = b_1 - b_2 - 0.$$

Since the left hand side is an element of N, this shows that $b_1 - b_2 \in N$.

In the other direction, suppose that $b_1, b_2 \in \mathrm{Ball}(B)$ with $b_1 - b_2 \in N$, so there exists $a \in A$ such that $b_1 - b_2 = f(a) - g(a)$. If $\|a\| > 1$, define $\gamma = \|a\|$ and $a' = \frac{a}{\gamma}$, otherwise define $\gamma = 1$ and $a' = a$. In either case we have $a' \in \mathrm{Ball}(A)$

and $a = \gamma a'$. Define $b_{3,1} = f(a')$ and $b_{3,2} = g(a')$, so $(b_{3,1}, b_{3,2}) \in R \subseteq S$, and $b_1 - b_2 = \gamma(b_{3,1} - b_{3,2})$.

Given $\beta \in [0, 1)$, we define

$$\begin{aligned} b_{1,1,\beta} &= \beta b_1 + (1 - \beta) b_{3,1} \\ b_{1,2,\beta} &= \beta b_1 + (1 - \beta) b_{3,2} \\ b_{2,2,\beta} &= \beta b_2 + (1 - \beta) b_{3,2}. \end{aligned}$$

Since S is convex and a reflexive relation, we have $(b_{1,1,\beta}, b_{1,2,\beta}) \in S$ for all $\beta \in [0, 1)$. We now find an alternative definition of $b_{1,2,\beta}$ in terms of $b_{1,1,\beta}$ in order to apply (4.5). We see that

$$\begin{aligned} b_{1,1,\beta} - b_{1,2,\beta} &= (1 - \beta)(b_{3,1} - b_{3,2}) && \text{and} \\ b_{1,2,\beta} - b_{2,2,\beta} &= \beta(b_1 - b_2) = \beta\gamma(b_{3,1} - b_{3,2}), \end{aligned}$$

so since $\beta < 1$ we have $b_{1,2,\beta} - b_{2,2,\beta} = \frac{\beta\gamma}{1-\beta}(b_{1,1,\beta} - b_{1,2,\beta})$. If we rearrange this using the fact that $0 \leq \beta < 1$ and $\gamma \geq 1$ we get

$$b_{1,2,\beta} = \frac{\beta\gamma}{1 - \beta + \beta\gamma} b_{1,1,\beta} + \frac{1 - \beta}{1 - \beta + \beta\gamma} b_{2,2,\beta},$$

so by defining $\alpha_\beta = \frac{\beta\gamma}{1-\beta+\beta\gamma}$, we have $b_{1,2,\beta} = \alpha_\beta b_{1,1,\beta} + (1 - \alpha_\beta) b_{2,2,\beta}$ where $\alpha_\beta \in [0, 1)$, for all $\beta \in [0, 1)$. Since S is a reflexive relation, $(b_{1,1,\beta}, \alpha_\beta b_{1,1,\beta} + (1 - \alpha_\beta) b_{2,2,\beta}) \in S$, so by (4.5), $(b_{1,1,\beta}, b_{2,2,\beta}) \in S$ for all $\beta \in [0, 1)$. By the continuity of addition and scalar multiplication, $b_1 = \lim_{\beta\to1} b_{1,1,\beta}$ and $b_2 = \lim_{\beta\to1} b_{2,2,\beta}$. Therefore $(b_1, b_2) \in S$, because S is closed. This concludes the proof that the equivalence relation defined by N agrees with S when restricted to $\mathrm{Ball}(B)$.

We now prove that N is closed. By what we have just proved, if $(b_1, b_2) \in S$, then $b_1 - b_2 \in N$, so $D \subseteq N$ and therefore $\mathrm{Ball}(B) \cap D \subseteq \mathrm{Ball}(B) \cap N$. To prove the converse inclusion, suppose $b \in \mathrm{Ball}(B) \cap N$. Then $b \in \mathrm{Ball}(B)$ with $b - 0 = b \in N$, so by what we proved in the previous paragraphs, $(b, 0) \in S$, so $b = b - 0 \in D$.

Since D is the continuous image of S, it is compact, so $\mathrm{Ball}(B) \cap N = \mathrm{Ball}(B) \cap D$ is compact, and therefore closed. By the Krein-Šmulian theorem [27, IV.6.4 Corollary], N is closed. Since the weak-* topology is coarser than the norm topology, N is also closed in the norm topology. It follows that the quotient mapping $q : B \to B/N$ is a coequalizer of (f, g) in both $\mathbf{W^*Alg}$ and $\mathbf{C^*Alg}$ (by Proposition 4.3), and if A, B are commutative then it is a coequalizer in both $\mathbf{CW^*Alg}$ and $\mathbf{CC^*Alg}$. We also have that $\mathrm{Ball}(q) : \mathrm{Ball}(B) \to \mathrm{Ball}(B/N)$ is the coequalizer of $(\mathrm{Ball}(f), \mathrm{Ball}(g))$ in $\mathbf{Set}$ because for all $b_1, b_2 \in \mathrm{Ball}(B)$:

$$q(b_1) = q(b_2) \Leftrightarrow b_1 - b_2 \in N \Leftrightarrow (b_1, b_2) \in S,$$

which is the equivalence relation we quotient by to define the coequalizer in $\mathbf{Set}$. □

Theorem 4.6. *The forgetful functors $U : \mathbf{W^*Alg} \to \mathbf{C^*Alg}$, $V = \mathrm{Ball} \circ U : \mathbf{W^*Alg} \to \mathbf{Set}$ and the restrictions $U : \mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$ and $V : \mathbf{CW^*Alg} \to \mathbf{Set}$ are monadic.*

Proof. By Corollary 2.6 U reflects isomorphisms. We also have that if $f : A \to B$ is a unital *-homomorphism of W*-algebras such that $\mathrm{Ball}(f)$ is an isomorphism in **Set**, and therefore bijective, then by rescaling elements to be in the unit ball we see that f is bijective and so is a *-isomorphism. This means that V reflects isomorphisms as well. By Proposition 4.4 it then follows from Barr and Wells's version of Beck's monadicity theorem [1, Theorem 3.3.14] that U and V are monadic, *i.e.* the comparison functor is an equivalence. □

We needed Barr and Wells's stronger version of Beck's monadicity theorem where we are allowed to assume the coequalizers are reflexive in order to ensure that N was a *-ideal.

We note the following corollary, because it seems to be previously unknown.

Corollary 4.7. *The categories* $\mathbf{W^*Alg}$ *and* $\mathbf{CW^*Alg}$ *are exact (in the sense of Barr).*

Proof. They are monadic over **Set**, and in **Set** every regular epimorphism splits, so by [22, Theorem 2.6] they are exact categories. □

5 A Comonad and a Monad

We start by looking for a left adjoint to the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$, to play the role of $C \circ \mathcal{S}$ in (2.21). If we are starting with an enveloping algebra A^{**} already, then for $B \in \mathbf{CW^*Alg}$:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) &\cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) \cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B)\\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B),\end{aligned} \tag{5.1}$$

and these isomorphisms are natural in B. The isomorphisms are probabilistic Gel'fand duality in the middle and the universal property of enveloping W*-algebras on either side. If all W*-algebras were isomorphic to enveloping algebras, we could finish here, but unfortunately this is not so. However, each W*-algebra A is canonically the quotient of A^{**}, and left adjoints preserve colimits so we pass the construction down from A^{**} along this quotient. It is possible that an argument similar to that used in [34, Example 14] could work, using the solution set condition, but the construction used here is more direct, as it involves only double dualization and reflexive coequalizers instead of general colimits.

Theorem 5.2. *The inclusion* $I : \mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ *has a left adjoint* F. *We write* $H = FI : \mathbf{CW^*Alg} \to \mathbf{CW^*Alg}$ *for the corresponding comonad. The coKleisli comparison morphism* $K_H : \mathcal{K}\ell(H) \to \mathbf{CW^*Alg}_{\mathrm{PU}}$ *is an isomorphism of categories.*

Proof. Let A be a unital commutative C*-algebra. We can find a commutative W*-algebra with the universal property for a left adjoint evaluated at A^{**} by the following isomorphisms, natural in $B \in \mathbf{CW^*Alg}$:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) &\cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) && \text{Definition 2.30}\\ &\cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B) && (2.21)\\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B) && \text{Definition 2.30}\end{aligned}$$

Write $\mathbf{CW^*Alg_{PU}}^0$ for the full subcategory of $\mathbf{CW^*Alg_{PU}}$ on double duals of unital commutative C*-algebras. Since for each $A^{**} \in \mathbf{CW^*Alg_{PU}}^0$ we have an object $C(S(A))^{**}$ with the universal property of a left adjoint to I, the proof (but not the statement) of the usual theorem relating different definitions of adjunction [20, IV.1 Theorem 2] proves in this case that we can build a functor $F^0 : \mathbf{CW^*Alg_{PU}}^0 \to \mathbf{CW^*Alg}$, with $F^0(A^{**}) = C(\mathcal{S}(A))$, and a natural isomorphism $\mathbf{CW^*Alg_{PU}}(\text{-},\text{-}) \cong \mathbf{CW^*Alg}(F^0(\text{-}),\text{-})$, as functors $(\mathbf{CW^*Alg_{PU}}^0)^{\mathrm{op}} \times \mathbf{CW^*Alg} \to \mathbf{Set}$.

Since $\mathbf{CW^*Alg}$ is monadic over $\mathbf{CC^*Alg}$, the canonical presentation of $A \in \mathbf{CW^*Alg}$

$$\begin{tikzcd} A^{****} \arrow[r, bend left, "\epsilon_{A^{**}}"] \arrow[r, shift right, "\epsilon_A^{**}"'] & A^{**} \arrow[l, shift right, "\eta_A^{**}"'] \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

is a reflexive coequalizer, where ϵ and η are the counit and unit from Definition 2.30 (this is essentially what was proved in Proposition 4.4 and Theorem 4.6).

Let $\mathcal{R}$ be the "walking reflexive pair", *i.e.* the category with two objects X, Y, two morphisms $f, g : X \to Y$ and one morphism $r : Y \to X$ such that $f \circ r = g \circ r = \mathrm{id}_Y$. Then for any category $\mathcal{C}$, the functor category $\mathcal{C}^{\mathcal{R}}$ is the category of reflexive pairs. Since $\mathbf{CW^*Alg}$ is monadic over $\mathbf{CC^*Alg}$, there is a functor $\mathrm{Pres} : \mathbf{CW^*Alg} \to \mathbf{CW^*Alg}^{\mathcal{R}}$ mapping each object to the reflexive pair from its canonical presentation, and taking a morphism $f : X \to Y$ to the pair of morphisms (f, f). For notation, we will simply write the triple of functions that form the reflexive pair, so $\mathrm{Pres}(A) = (\epsilon_A^{**}, \epsilon_{A^{**}}, \eta_A^{**})$. It is apparent that the image of Pres lies in $\mathbf{CW^*Alg_{PU}}^0$, because each object is a double dual. We write Coeq for the functor mapping reflexive pairs to their coequalizers, both for the one $\mathbf{CW^*Alg}^{\mathcal{R}} \to \mathbf{CW^*Alg}$ and the one from the full subcategory of $\mathbf{CW^*Alg_{PU}}^{\mathcal{R}}$ on reflexive pairs from $\mathbf{CW^*Alg}^{\mathcal{R}}$ (which exists because of Proposition 4.3). In the following proof we will also write Eq for the functor $\mathbf{Set}^{\mathcal{R}} \to \mathbf{Set}$ that maps a reflexive pair to its equalizer.

We define $F : \mathbf{CW^*Alg_{PU}} \to \mathbf{CW^*Alg}$ by $F = \mathrm{Coeq} \circ (F^0)^{\mathcal{R}} \circ \mathrm{Pres}$. Then we have

$$\begin{aligned}
\mathbf{CW^*Alg}(F(A), B) &= \mathbf{CW^*Alg}(\mathrm{Coeq}(F^0(\epsilon_A^{**}), F^0(\epsilon_{A^{**}}), F^0(\eta_A^{**})), B) \\
&\cong \mathrm{Eq}(\mathbf{CW^*Alg}(F^0(\epsilon_A^{**}), B), \mathbf{CW^*Alg}(F^0(\epsilon_{A^{**}}), B), \mathbf{CW^*Alg}(F^0(\eta_A^{**}), B)) \\
&\cong \mathrm{Eq}(\mathbf{CW^*Alg_{PU}}(\epsilon_A^{**}, I(B)), \mathbf{CW^*Alg_{PU}}(\epsilon_{A^{**}}, I(B)), \mathbf{CW^*Alg_{PU}}(\eta_A^{**}, I(B))) \\
&\cong \mathbf{CW^*Alg_{PU}}(\mathrm{Coeq}(\epsilon_A^{**}, \epsilon_{A^{**}}, \eta_A^{**}), I(B)) \\
&\cong \mathbf{CW^*Alg_{PU}}(A, I(B)).
\end{aligned}$$

naturally for $A \in \mathbf{CW^*Alg_{PU}}$ and $B \in \mathbf{CW^*Alg}$.

Therefore $H = FI$ is a comonad on $\mathbf{CW^*Alg}$. The coKleisli comparison functor $K_H : \mathcal{K}\ell(H) \to \mathbf{CW^*Alg_{PU}}$ is given on objects by I, which is the identity, and on maps it is given by the isomorphism $\mathcal{K}\ell(H)(A, B) = \mathbf{CW^*Alg}(FI(A), B) \cong \mathbf{CW^*Alg_{PU}}(I(A), I(B)) = \mathbf{CW^*Alg_{PU}}(A, B)$. So this is an isomorphism of categories. □

The observation that an adjunction has an isomorphism for a coKleisli comparison functor iff the right adjoint functor is surjective was pointed out to the author by Bram Westerbaan [34, Theorem 9].

Theorem 5.3. *There is a monad $T : \mathbf{Meas} \to \mathbf{Meas}$ such that $\mathcal{K}\ell(T) \simeq \mathbf{CW^*Alg_{PU}}^{\mathrm{op}}$.*

Proof. Combine Theorem 5.2 with Gel'fand duality for commutative W*-algebras (Subsection 2.5), and the comonad H is mapped to a monad T. We also get an equivalence $\mathcal{K}\ell(T) \cong \mathcal{K}\ell(H)^{\mathrm{op}} \cong \mathbf{CW^*Alg_{PU}}^{\mathrm{op}}$. □

If necessary, we can evaluate the formulas implicit in the construction given in Theorem 5.2 and describe $T(X)$ as a particular clopen subset of $\mathrm{Spec}(C(S(A))^{**})$, but we omit this for reasons of space.

To relate this to what we said in the abstract, we have that $\mathbf{Meas}(1, T(X)) \cong \mathbf{CW^*Alg}(H(L^\infty(X)), L^\infty(1)) \cong \mathbf{CW^*Alg_{PU}}(L^\infty(X), \mathbb{C})$. The latter is the set of normal states on $L^\infty(X)$, which by the Radon-Nikodym theorem is the image of the probability density functions (positive L^1 functions with integral 1) under the embedding of $L^1(X)$ in $L^\infty(X)$.

Be warned that $\mathbf{Meas}(1, Y)$ is not the same as the underlying set of Y. If all singletons are measurable, it corresponds to the non-null singletons of Y, *i.e.* the singletons of strictly positive measure.

We can also give a description of $T(X)$ for X a countable set with the counting measure.

Proposition 5.4. *For any countable set X with at least two elements, equipped with the counting measure, we have*

$$T(X) \cong \mathrm{Spec}(C(2^\omega)^{**}) \cong (2^\omega, \mathcal{P}(2^\omega), \nu_d) + (2^\omega \times 2^\omega, \mathcal{P}(2^\omega) \otimes \widehat{\mathcal{B}o(2^\omega)}, \nu_d \otimes \hat{\nu}_c)$$

with the definitions for the measure space on the right hand side as in Theorem 3.9. If X is a singleton, then $T(X) \cong 1$, and $T(\emptyset) = \emptyset$.

Proof. For X a finite set, we have $L^\infty(X) \cong \mathbb{C}^X$, which is finite-dimensional, so $L^\infty(X) \cong L^\infty(X)^{**}$ under the evaluation mapping. Therefore $L^\infty(X)$ is in the category $\mathbf{CW^*Alg_{PU}}^0$ of double dual commutative W*-algebras, from the proof of Theorem 5.2, so $F(L^\infty(X)) \cong F^0(L^\infty(X)) = C(\mathcal{S}(L^\infty(X)))^{**}$. Now, $\mathcal{S}(L^\infty(X))$ is the simplex with $|X|$ vertices. If $|X| \geq 2$, this is an uncountable compact metrizable space, so $F(L^\infty(X)) \cong C(2^\omega)^{**}$ by Theorem 3.9, and therefore $T(X) \cong (2^\omega, \mathcal{P}(2^\omega), \nu_d) + (2^\omega \times 2^\omega, \mathcal{P}(2^\omega) \otimes \widehat{\mathcal{B}o(2^\omega)}, \nu_d \otimes \hat{\nu}_c)$. If $|X| = 1$, then $\mathcal{S}(L^\infty(X))$ is a singleton, so $C(\mathcal{S}(L^\infty(X))) \cong C(1) = L^\infty(1)$ and $T(X) \cong 1$. If $X = \emptyset$, then $\mathcal{S}(L^\infty(X)) = \emptyset$ and $C(\mathcal{S}(L^\infty(X))) = C(\emptyset) = L^\infty(\emptyset)$. Since the only way to have $L^\infty(X) \cong L^\infty(\emptyset)$ is for X to actually be empty[12], we have $T(X) = \emptyset$.

For X countably infinite, we recall the space $c \subseteq \ell^\infty(\mathbb{N})$ of convergent sequences, which is a C*-subalgebra of $\ell^\infty(\mathbb{N})$, and we have the isomorphism $c^{**} \cong \ell^\infty(\mathbb{N})$, known since the time of Banach. Therefore $L^\infty(X)$ is also contained in $\mathbf{CW^*Alg_{PU}}^0$ (up to isomorphism), so $F(L^\infty(X)) \cong F^0(L^\infty(X)) \cong C(\mathcal{S}(c))^{**}$. Since c is separable, $\mathcal{S}(c)$ is metrizable (and compact) in the weak-* topology, and it is uncountable because it is a convex set containing more than two points. Therefore $C(\mathcal{S}(c))^{**} \cong C(2^\omega)^{**}$ and we get the same result as in the case of a finite set of at least two elements. □

[12] The definition of strictly localizable measure space that the author uses does not allow a space all of whose measurable subsets have measure zero. One has to make such a convention to get an equivalence involving functions between measure spaces.

Acknowledgements

Many thanks to Bram Westerbaan, Sam Staton, Phil Scott, Bart Jacobs and Tobias Fritz and Kenta Cho.

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A Extra Statements Needed for Products and Coequalizers of W*-algebras

In this section we give extra proofs needed to show that the product of C*-algebras, applied to W*-algebras gives the product of W*-algebras, and the predual is the ℓ^1-direct sum of the preduals. We also describe the quotient norm relative to a closed subspace of a Banach space.

Let $\mathbf{Ban}_1$ be the category with Banach spaces (over $\mathbb{C}$) as objects and contractions (linear maps with operator norm ≤ 1) as morphisms.

Definition A.1. *Let $(E_i)_{i\in I}$ be a family of Banach spaces. The ℓ^∞-*direct sum *of $(E_i)_{i\in I}$ has underlying space the uniformly bounded families of elements of $(E_i)_{i\in I}$, which is to say:*

$$\prod_{i\in I} E_i = \{(x_i)_{i\in I} \mid x_i \in E_i \text{ and } \exists \alpha \in \mathbb{R}_{\geq 0}. \forall i \in I. \|x_i\|_{E_i} \leq \alpha\}.$$

The vector space operations are defined pointwise and the norm is:

$$\|(x_i)_{i\in I}\|_{\prod_{i\in I} E_i} = \sup_{i\in I} \|x_i\|_{E_i}.$$

This is a Banach space.

The maps $\pi_i : \prod_{i\in I} E_i \to E_i$ defined by

$$\pi_i((x_j)_{j\in I}) = x_i$$

are contractions. If D is a Banach space, and $(g_i)_{i\in I}$ a family of contractions with $g_i : D \to E_i$, then there is a unique contraction $\langle g_i\rangle_{i\in I} : D \to \prod_{i\in I} E_i$ such that for all $i \in I$, $\pi_i \circ \langle g_j\rangle_{j\in I} = g_i$, which is defined by

$$\langle g_i\rangle_{i\in I}(d) = (g_i(d))_{i\in I}.$$

So this defines the products in $\mathbf{Ban}_1$*.*

Proof. To prove that $\prod_{i\in I} E_i$ is a normed space, it is easiest to show that the set-theoretic product is a vector space, and the norm $\|\text{-}\|$ above can be defined on it but takes infinite values. The space $\prod_{i\in I} E_i$ is the subspace where the norm is finite. This proves that $\prod_{i\in I} E_i$ is a vector space and $\|\text{-}\|$ is a norm at the same time.

It is immediate from the definitions that each π_i is linear and a contraction. We prove that $\prod_{i\in I} E_i$ is a Banach space as follows. Let $(x_{i,n})_{i\in I,n\in\mathbb{N}}$ be a Cauchy sequence. Then for each $j \in I$, $(x_{j,n})_{n\in\mathbb{N}}$ is Cauchy because π_j is a contraction, so there exists $y_j \in E_j$ such that $(x_{j,n})_{n\in\mathbb{N}} \to y_j$. We aim to show that $(x_{i,n})_{i\in I,n\in\mathbb{N}} \to (y_i)_{i\in I}$.

Let $\epsilon > 0$, and use the Cauchyness of $(x_{i,n})_{i\in I,n\in\mathbb{N}}$ take an $N \in \mathbb{N}$ such that for all $n, m \geq N$, for all $i \in I$ we have $\|x_{i,n} - x_{j,n}\| \leq \frac{\epsilon}{2}$. Suppose for a contradiction that there exists $n \geq N$ and $i \in I$ such that $\|x_{i,n} - y_i\| \geq \epsilon$. As $(x_{i,n})_{n\in\mathbb{N}} \to y_i$, there exists N' such that for all $m \geq N'$, $\|x_{i,m} - y_i\| < \frac{\epsilon}{2}$. Taking $m \geq \max\{N, N'\}$, we have

$$\|x_{i,n} - y_i\| \leq \|x_{i,n} - x_{i,m}\| + \|x_{i,m} - y_i\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

a contradiction. Therefore $(x_{i,n})_{i\in I,n\in\mathbb{N}} \to (y_i)_{i\in I}$.

Finally, let D be a Banach space, and $(g_i)_{i\in I}$ a family of contractions with $g_i : D \to E_i$. Given the definition of $\langle g_i\rangle_{i\in I} : D \to \prod_{i\in I} E_i$, we need to show that it actually defines a contraction. So with $d \in D$:

$$\|\langle g_i\rangle_{i\in I}(d)\| = \sup_{i\in I} \|g_i(d)\| \leq \sup_{i\in I} \|d\| = \|d\|,$$

because $\|g_i\| \leq 1$ for all $i \in I$. It is easy to verify that $\langle g_i\rangle_{i\in I}$ is linear, so this shows that it is defined and a contraction. The uniqueness part of the universal property is immediate from the definition. □

In the following case, and once again later, it is convenient to prove completeness using the following lemma. There are various versions of it in the literature, of different strengths and with various different assumptions. In my own work, I first used a version of it in [12, Lemma 2.2.15], where I had been badly unable to prove certain facts without it. It often occurs without attribution, and may date back to Riesz or Banach, or even earlier.

Lemma A.2. *A normed space E is complete iff* $\mathrm{Ball}(E)$ *is σ-convex.*

Proof. It is not hard to show that the partial sums of a σ-convex combination form a Cauchy sequence, so $\mathrm{Ball}(E)$ for E a Banach space is σ-convex. To prove the converse, suppose that $\mathrm{Ball}(E)$ is σ-convex, and let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in E. We first observe that Cauchyness implies that to prove that $(x_i)_{i\in\mathbb{N}}$ converges it suffices to prove that a subsequence converges (to the same limit). So we replace $(x_i)_{i\in\mathbb{N}}$ with a subsequence such that $\|x_{i+1} - x_i\| < 2^{-i}$ for all $i \in \mathbb{N}$.

Define $y_1 = 2x_1$ and $y_{i+1} = 2^{i+1}(x_{i+1} - x_i)$. We have $\|y_{i+1}\| < 2^{i+1}2^{-i} = 2$ and $\|y_1\| = 2\|x_1\|$, so the sequence $(y_i)_{i\in\mathbb{N}}$ is contained in $n\mathrm{Ball}(E)$ for some $n \in \mathbb{N}$. By continuity of multiplication, $n\mathrm{Ball}(E)$ is σ-convex, so we can define $x = \sum_{i=1}^{\infty} 2^{-i} y_i \in E$. Then

$$\sum_{i=1}^{m} 2^{-i} y_i = \frac{1}{2} 2x_1 + \sum_{i=1}^{m-1} 2^{-(i+1)} 2^{i+1}(x_{i+1} - x_i) = x_1 + \sum_{i=1}^{m-1}(x_{i+1} - x_i) = x_m,$$

because the sum telescopes. Therefore $x_i \to x$, so E is complete. $\square$

Definition A.3. *The* ℓ^1-direct sum of $(E_i)_{i\in I}$ *has underlying space the families whose norms form a summable series, which is to say:*

$$\bigoplus_{i\in I} E_i = \left\{ (x_i)_{i\in I} \;\middle|\; x_i \in E_i \text{ and } \sum_{i\in I} \|x_i\|_{E_i} < \infty \right\}.$$

The vector space operations are defined pointwise and the norm is:

$$\|(x_i)_{i\in I}\|_{\bigoplus_{i\in I} E_i} = \sum_{i\in I} \|x_i\|_{E_i}.$$

This is a Banach space.

The maps $\kappa_i : E_i \to \bigoplus_{i\in I} E_i$ *defined by*

$$\kappa_i(x)_j = \begin{cases} x & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

are contractions. If F is a Banach space, and $(g_i)_{i\in I}$ a family of contractions with $g_i : E_i \to F$, then there is a unique contraction $[g_i]_{i\in I} : \bigoplus_{i\in I} E_i \to F$ such that for all $j \in I$, $[g_i]_{i\in I} \circ \kappa_j = g_j$, which is defined by

$$[g_i]_{i\in I}((x_j)_{j\in I}) = \sum_{i\in I} g_i(x_i).$$

So this defines the coproducts in $\mathbf{Ban}_1$.

Proof. As in the proof of Definition A.1, consider $\bigoplus_{i\in I} E_i$ as a subset of the set-theoretic product, and consider the norm as being allowed to take the value ∞ and $\bigoplus_{i\in I} E_i$ the subset on which it is finite-valued. Then it is easy to show that $\|\text{-}\|$ is a norm and $\bigoplus_{i\in I} E_i$ is a vector space at the same time.

We use Lemma A.2 to show that $\bigoplus_{i\in I} E_i$ is complete by showing that the unit ball is σ-convex. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_{\geq 0}$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$, and take $(x_{i,n})_{i\in I, n\in\mathbb{N}}$ to be a sequence in $\mathrm{Ball}\left(\bigoplus_{i\in I} E_i\right)$. We aim to show that $\sum_{n=1}^{\infty} \alpha_n (x_{i,n})_{i\in I}$ converges.

First, since for each $n \in \mathbb{N}$, $\|(x_{i,n})_{i\in I}\| \leq 1$, it also holds that for each $i \in I$, $\|x_{i,n}\|_{E_i} \leq 1$. So by Lemma A.2, for all $i \in I$ the sum $\sum_{n=1}^{\infty} \alpha_i x_{i,n}$ converges to some element $y_i \in \mathrm{Ball}(E_i)$.

We have

$$\begin{aligned}
\|(y_i)_{i\in I}\| &= \sum_{i\in I} \|y_i\|_{E_i} \\
&= \sum_{i\in I} \left\| \sum_{n=1}^{\infty} \alpha_n x_{i,n} \right\| \\
&\le \sum_{i\in I} \sum_{n=1}^{\infty} \alpha_n \|x_{i,n}\|_{E_i} \\
&= \sum_{n=1}^{\infty} \alpha_n \sum_{i\in I} \|x_{i,n}\|_{E_i} \\
&= \sum_{n=1}^{\infty} \alpha_n \|(x_{i,n})_{i\in I}\| \\
&\le \sum_{n=1}^{\infty} \alpha_n = 1.
\end{aligned}$$

where we have freely used the continuity of the norm and various facts about rearranging sums of nonnegative reals. Therefore $(y_i)_{i\in I} \in \mathrm{Ball}\left(\bigoplus_{i\in I} E_i\right)$. So it remains to show that $\sum_{n=1}^{\infty} \alpha_i (x_{i,n})_{i\in I} = (y_i)_{i\in I}$.

Let $\epsilon > 0$, and take $N \in \mathbb{N}$ such that for all $m \ge N$ such that $1 - \sum_{n=1}^{m} \alpha_n < \epsilon$. Then for all $m \ge N$

$$\begin{aligned}
\left\| (y_i)_{i\in I} - \sum_{n=1}^{m} \alpha_n (x_{i,n})_{i\in I} \right\| &= \sum_{i\in I} \left\| \sum_{n=1}^{\infty} \alpha_n x_{i,n} - \sum_{n=1}^{m} \alpha_n x_{i,n} \right\| \\
&= \sum_{i\in I} \left\| \sum_{n=m+1}^{\infty} \alpha_n x_{i,n} \right\| \\
&\le \sum_{i\in I} \sum_{n=m+1}^{\infty} \alpha_n \|x_{i,n}\|_{E_i} \\
&= \sum_{n=m+1}^{\infty} \alpha_n \sum_{i\in I} \|x_{i,n}\|_{E_i} \\
&= \sum_{n=m+1}^{\infty} \alpha_n \|(x_{i,n})_{i\in I}\| \\
&\le \sum_{n=m+1}^{\infty} \alpha_n < \epsilon.
\end{aligned}$$

Therefore $\bigoplus_{i\in I} E_i$ is complete by Lemma A.2.

It is easy to show that for all $i \in I$, κ_i is a contraction. So let F be a Banach space and $(g_i)_{i\in I}$ a family of contractions $g_i : E_i \to F$. We must first show that for all $(x_i)_{i\in I} \in \bigoplus_{i\in I} E_i$, the sum defining $\langle g_i \rangle_{i\in I}((x_i)_{i\in I})$ converges. We see

$$\sum_{i\in I} \|g_i(x_i)\| \le \sum_{i\in I} \|g_i\| \cdot \|x_i\|_{E_i} \le \sum_{i\in I} \|x_i\|_{E_i} = \|(x_i)_{i\in I}\|,$$

so the sum is absolutely convergent, and therefore converges in the Banach space F. The same inequality will show that $\langle g_i \rangle_{i\in I}$ is a contraction, once we know that it is linear. It is easy to prove linearity using linearity of the maps $(g_i)_{i\in I}$ and the ability to rearrange an absolutely convergent sum. It is clear that for all $j \in I$, $\langle g_i\rangle_{i\in I} \circ \kappa_j = g_j$. If $h : \bigoplus_{i\in I} E_i \to F$ is a contraction such that for all $j \in I$, $h \circ \kappa_j = g_j$, then it is easy to show that h and $\langle g_i\rangle_{i\in I}$ agree on sequences of finite support, and since these are norm-dense $h = \langle g_i\rangle_{i\in I}$. □

Definition A.4. *Let F be a Banach space and $E \subseteq F$ a closed subspace. Define the seminorm $\|\text{-}\|_{\delta(E)} : F \to \mathbb{R}_{\geq 0}$ to be the distance from E,* i.e.

$$\|y\|_{\delta(E)} = \inf\{\|y - x\|_F \mid x \in E\}.$$

Define $G = F/E$ and $p : F \to G$ to be the quotient map. Then $\|\text{-}\|_G$ defined by

$$\|p(y)\|_G = \|y\|_{\delta(E)},$$

is a well-defined norm on G making it into a Banach space and p into a contraction. We also have that for each $z \in G$, for all $\epsilon > 0$, there is a $y \in F$ such that $p(y) = z$ and $\|y\|_F < \|z\|_G + \epsilon$.

Given a vector space H and a linear map $f : F \to H$ such that $E \subseteq f^{-1}(0)$, we can define $\tilde{f} : F/E \to H$ by

$$\tilde{f}([x]) = f(x),$$

which is a well-defined linear map and the unique function such that $\tilde{f} \circ q = f$. If H is a normed space and f is bounded, then $\|\tilde{f}\| \leq \|f\|$ and is therefore bounded.

Proof. We first prove that $\|\text{-}\|_{\delta(E)}$ is a seminorm. To show that it does not take the value ∞, we observe that as $0 \in E$, for all $y \in F$ we have $\|y\|_{\delta(E)} \leq \|y\|_F$.

If $\alpha = 0$ and $y \in F$, we have $\|\alpha y\|_{\delta(E)} = \|0\|_{\delta(E)} = 0 = 0\|y\|_{\delta(E)}$, because $0 \in E$. If $\alpha \in k$ and $\alpha \neq 0$, then

$$\begin{aligned}
\|\alpha y\|_{\delta(E)} &= \inf\{\|\alpha y - x\|_F \mid x \in E\} \\
&= \inf\{\alpha\|y - \alpha^{-1}x\|_F \mid x \in E\} \\
&= \alpha \inf\{\|y - x\|_F \mid x \in \alpha^{-1}E\} \\
&= \alpha \inf\{\|y - x\|_F \mid x \in E\} \\
&= \alpha\|y\|_{\delta(E)}.
\end{aligned}$$

Now let $y_1, y_2 \in F$. For all $\epsilon > 0$, there exist $x_1, x_2 \in E$ such that $\|y_i - x_i\|_F \leq \|y_i\|_{\delta(E)} + \frac{\epsilon}{2}$, where $i \in \{1, 2\}$. Since $x_1 + x_2 \in E$, we have

$$\begin{aligned}
\|y_1 + y_2\|_{\delta(E)} &\leq \|y_1 + y_2 - (x_1 + x_2)\|_F \leq \|y_1 - x_1\|_F + \|y_2 - x_2\|_F \\
&\leq \|y_1\|_{\delta(E)} + \|y_2\|_{\delta(E)} + \epsilon.
\end{aligned}$$

Since this holds for all $\epsilon > 0$, we have $\|y_1 + y_2\|_{\delta(E)} \leq \|y_1\|_{\delta(E)} + \|y_2\|_{\delta(E)}$, completing the proof that $\|\text{-}\|_{\delta(E)}$ is a seminorm.

As an intermediate step, we show that $\|y\|_{\delta(E)} = 0$ iff $y \in E$. Clearly if $y \in E$ then $\|y\|_{\delta(E)} = 0$. For the other direction, suppose that $\|y\|_{\delta(E)} = 0$. Then for all $i \in \mathbb{N}$, there exists $x_i \in E$ such that $\|y - x_i\|_F < 2^i$. Therefore $x_i \to y$, and so $y \in E$ because E is closed.

Now take $G = F/E$ as a vector space and $p : F \to G$ the linear quotient map. Define $\|p(y)\|_G = \|y\|_{\delta(E)}$. Suppose that $p(y_1) = p(y_2)$, so $y_2 - y_1 \in E$. Then

$$\begin{aligned} \|p(y_1)\|_G = \|y_1\|_{\delta(E)} &= \inf\{\|y_1 - x\|_F \mid x \in E\} \\ &= \inf\{\|y_2 - (y_2 - y_1) + x\|_F \mid x \in E\} \\ &= \|y_2\|_{\delta(E)} = \|p(y_2)\|_G, \end{aligned}$$

proving it is well-defined. The fact that it is a seminorm follows from $\|\text{-}\|_{\delta(E)}$ being a seminorm, and it is a norm because $\|\text{-}\|_{\delta(E)}$ vanishes precisely on E. We also have that p is a contraction because for all $y \in F$, $\|p(y)\|_G = \|y\|_{\delta(E)} \leq \|y\|_F$.

Given $y' \in F$, and $\epsilon > 0$, we can find $y \in F$ such that $y - y' \in E$ and $\|y\|_F \leq \|y'\|_{\delta(E)} + \epsilon$ by observing that there exists $x \in E$ such that $\|y' - x\|_F \leq \|y'\|_{\delta(E)} + \epsilon$, and we can therefore just take $y = y' - x$. It follows immediately that for each $z \in G$ and $\epsilon > 0$, there is a $y \in F$ such that $p(y) = z$ and $\|y\|_F \leq \|z\|_G + \epsilon$.

To show that G is complete, we show that $\mathrm{Ball}(G)$ is σ-convex and use Lemma A.2. Let $(z_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Ball}(G)$ and $(\alpha_i)_{i\in\mathbb{N}}$ a sequence in $\mathbb{R}_{\geq 0}$ that is summable to 1. Fix some $\epsilon > 0$, and observe that for all $i \in \mathbb{N}$, we can find $y_i \in F$ such that $p(y_i) = z_i$ and $\|y_i\|_F \leq \|z_i\|_G + \epsilon$. Since F is a Banach space, the sum $\sum_{i=1}^{\infty} \alpha_i y_i$ converges to some $y \in F$. Define $z = p(y)$, and observe that by linearity and continuity of p, $\sum_{i=1}^{\infty} \alpha z_i = z$, and that if we chose different representatives $(y_i)_{i\in\mathbb{N}}$ we still get the same z. Since $\epsilon > 0$ is arbitrary, this implies that $z \in (1+\epsilon)\mathrm{Ball}(G)$ and therefore $z \in \mathrm{Ball}(G)$. Therefore G is complete, as it has a σ-convex unit ball.

Let H be a vector space and $f : F \to H$ a linear map such that $E \subseteq f^{-1}(0)$. If $[x_1] = [x_2]$, then $x_1 - x_2 \in E$, so

$$f(x_2) = f(x_2) + 0 = f(x_2) + f(x_1 - x_2) = f(x_2 + x_1 - x_2) = f(x_1),$$

and therefore $\tilde{f}$ is well defined. It is easy to show that it is linear by using the linearity of q. We also have that $\tilde{f}$ is the unique function $F/E \to H$ such that $\tilde{f} \circ q = f$, essentially by definition.

If H is normed and f is bounded, then for all $y \in F/E$ and $\epsilon > 0$ there exists $x \in F$ with $q(x) = y$ and $\|x\|_F \leq \|y\|_G + \epsilon$, so

$$\|\tilde{f}(y)\|_H = \|f(x)\|_H \leq \|f\|\|x\|_F \leq \|f\|(\|y\|_G + \epsilon).$$

By letting $\epsilon \to 0$, we get $\|\tilde{f}(y)\|_H \leq \|f\|\|y\|_G$, and therefore $\tilde{f}$ is bounded with operator norm $\|\tilde{f}\| \leq \|f\|$. □

Definition A.5. *Define $\mathscr{F} : \mathbf{Ban}_1 \to \mathbf{Ban}_1^{\mathrm{op}}$ as follows. On objects $\mathscr{F}(E) = E^*$, while on a contraction $f : E \to F$, we define*

$$\mathscr{F}(f)(\psi) = \psi \circ f,$$

where $\psi \in F^*$. Define $\mathscr{G} : \mathbf{Ban}_1^{\mathrm{op}} \to \mathbf{Ban}_1$ to be $\mathscr{F}^{\mathrm{op}}$. The evaluation mapping $\mathrm{ev} : E \to E^{**}$ *defines both a unit $\eta : \mathrm{Id} \Rightarrow \mathscr{G}\mathscr{F}$ and a counit $\epsilon : \mathscr{F}\mathscr{G} \Rightarrow \mathrm{Id}$ (interpreted in $\mathbf{Ban}_1^{\mathrm{op}}$ the second time, reversing the direction) making $\mathscr{F}$ a left adjoint to $\mathscr{G}$.*

Proof. Proving that $\mathscr{F}$ and $\mathscr{G}$ are functors is simple, and η and ϵ can be proved to be natural by the same proof which simply expands the definitions. Once the arrows are turned round to all be in $\mathbf{Ban}_1$, the two triangles defining an adjunction [20, IV.1 Theorem 2 (v)] are equivalent. Like the case of naturality, it is easily proved by expanding the definitions. $\square$

Theorem A.6. *If $(E_i)_{i\in I}$ is a family of Banach spaces, then $\left(\bigoplus_{i\in I} E_i\right)^*$ is a product of this family with $(\kappa_i^*)_{i\in I}$ as projections. If $f, g : E \to F$ is a pair of contractions, and $(G, p : F \to G)$ their coequalizer, then (G^*, p^*) is an equalizer of (f^*, g^*).*

Proof. Apply the fact that left adjoints preserve colimits to $\mathscr{F}$. $\square$

Observe that the fact that $\mathscr{G}$ preserves limits proves exactly the same facts. The space $(\ell^\infty)^*$ is not separable and therefore not isomorphic to ℓ^1, so the dual of a product is not the coproduct of the duals.